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MODELING AND ADJUSTING GLOBAL OCEAN TIDES
USING SEASAT ALTIMETER DATA

Georges Blaha

Nova University Oceanographic Center
8000 North Ocean Drive
Dania, Florida 33004

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In the recent past, SEASAT altimeter data together with global geoidal parameters and sets of state vectors parameters have been adjusted at AFGL through the short-arc algorithm using NSWC precise ephemeris. The excellent quality of the data system, including the altimeter measurements, the ephemeris, the input geoidal parameters, etc., has been confirmed by the finding that the empirical variance for geoid undulations is significantly

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) lower than the theoretical variance. The altimeter residuals from this adjustment have served in regional modeling of short-wavelength geoidal features, as well as in studying geophysical phenomena such as ocean bottom topography. Since the geoid has been assumed to coincide with the ocean surface sensored by the altimeter, the sea-surface effects have been ignored in this adjustment.

Due to increasing geophysical interest in a realistic representation of the open ocean tide, the latest development of the short-arc satellite altimetry model allows for the inclusion of the most important tidal constituents. In particular, an adjustment algorithm has been designed in which four long-period constituents, three diurnal constituents and four semidiurnal constituents may be subject to adjustment within the overall adjustment of SEASAT altimetric observations. Except for the long-period constituents where the phase angle is considered fixed, both tidal amplitude and phase angle are adjustable. In attributing a priori weights to the tidal parameters, the aliasing of certain constituent frequencies into lower frequencies has to be considered.

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1. INTRODUCTION

The on-going effort whose various facets have been described e.g. in [Blaha, 1975, 1977, 1979, 1980, 1981] is concerned with the adjustment of satellite altimeter data in two steps, the first performed in terms of a truncated set of spherical-harmonic (S.H.) potential coefficients and the second performed in terms of point-mass (P.M.) magnitudes as parameters. The emphasis has been shifted recently toward SEASAT altimeter data as the main source of observed quantities in the first adjustment; gravity anomalies and other sources of geopotential information have been included via the weighted S.H. coefficients. At this stage (first adjustment), six weighted state vector (s.v.) parameters per orbital arc are also included in the simultaneous least-squares process. The first adjustment is global in character. One of its most important products is a revised set of S.H. coefficients which may be of interest in itself, and which is especially useful in predicting geoid undulations, gravity anomalies and other quantities related to the disturbing potential (such as deflections of the vertical or gravity gradients) on the global scale.

The data for the second adjustment consist mainly of the residuals from the first adjustment, although other quantities (gravity anomalies, deflections of the vertical, etc.) can enter this phase independently. In this process a more detailed, but regional, geoid is derived. Predictions of the other quantities just mentioned can also be made in the region of interest. A given set of point masses has a chosen distribution which, as a rule, is uniform and is characterized by the 1.6:1 depth-side ratio.

Denser sets of point masses could be superimposed on the basic set, leading to an even more detailed description of the gravity field in specific subregions. This can be related to the mean values of geoid undulations derived, for blocks of a certain size, from satellite altimetry. In some areas of pronounced and varied geoidal relief, such as in the Puerto Rico trench area, the differences between the mean and the actual undulations could become large and, thus, smaller blocks might be chosen to describe the geoid. From geoid undulations (or their means) one could derive other quantities related to the disturbing potential, upon using the appropriate cross-covariance functions. The present approach with point masses circumvents, by construction, the need for these functions.

One final product obtained with the P.M. adjustment superimposed on the S.H. adjustment is a set of contour maps based on predicted values for geoid undulations, gravity anomalies, etc. A typical example is a geoidal map in a region containing point masses. This region exhibits detailed geoidal features, while far from it the geoid described by the potential coefficients alone is very smooth; the transition from one region to the other is gradual.

Since the global S.H. adjustment is the basis for any detailed geoidal resolution, it has been refined on several occasions, both from the accuracy and the economy standpoints. For example, criteria have been established specifying the maximum and the minimum allowable length of satellite arcs. In terms of SEASAT altimetry these criteria translate into seven minutes and about one minute in duration, respectively, as described in Chapter 2 of [Blaha, 1981]. The same report contains also the development resulting in a substantial improvement in the economy of the short-arc algorithm, achieved through a reduction in the number of S.H. potential coefficients entering the orbital integrator. The corresponding reduction in the run-time requirements does not compromise the quality of the altimetric information. Indeed, the information represented by the SEASAT altimeter data, the NSWC precise ephemeris, the GEM 10 S.H. potential coefficients (at least within the 16,16 truncated set), as well as the reference field parameters has been found to be of excellent quality. The pertinent analysis is presented in Appendix 1 which draws a tentative conclusion with regard to the closed-form expression giving the degree variances for geoid undulations.

Due to the high quality of the SEASAT altimetric system compared to its predecessors (e.g., the GEOS-3 system including the broadcast ephemeris), modeling errors caused by certain sea level changes can no longer be ignored. The most important changes are those caused by the tide-generating forces of the moon and the sun. Of these, eleven long-period, diurnal and semidiurnal tidal effects are discussed in the body of this report, starting with the equilibrium tide. The initial development overlaps, to a certain extent, with Sections 3.1 and 3.2 of [Blaha, 1981] where, however, fewer tidal constituents were treated.

In addition to this Final Report, the reports covering the entire period of the present research contract have been [Blaha 1979, 1980, 1981].

2. EQUILIBRIUM TIDE

When determining the height of the theoretical, or equilibrium, tide, its individual component "h_j" is associated with the tidal constituent "j" of amplitude A_j and argument α_j. The total height is then the sum of the individual h_j's. The basic formulas adopted in this development are (129) - (131) of [USCGS,1958] abbreviated here as [US]. Since only the average values of the h_j's with regard to the longitude of the moon's mode are sought at a first stage, the "node factor", f, is taken as unity in the pertinent formulas. This implies, for example, that the "permanent tide" symbolized by A_0 is considered through its mean effect over one full revolution (or several full revolutions) of the moon's node; such a revolution is completed in about 18.6 years.

In considering the average (in the above sense) combined effect of the moon and the sun, the constituent height can be expressed by

$$h_j = A_j \cos \alpha_j . \tag{2.1}$$

The amplitude varies with ϕ, the geocentric latitude, as

$$A_j \simeq K_j \, G_j(\phi) \, C_j ; \tag{2.2}$$

here $f \equiv 1$ is assumed so that the coefficient C_j represents the mean value of a pertinent function with respect to the longitude of the moon's node. In the following, three indices (a, b, c) will be used:

a... long-period constituents,
b... diurnal constituents,
c... semidiurnal constituents.

The meaning of the symbols K_j and $G_j(\phi)$ is thus narrowed down to

$$K_a = \tfrac{1}{2}Ga \simeq 0.13334\,\text{m}\,, \tag{2.3a}$$

$$K_b = K_c = Ga \simeq 0.2667\,\text{m}\,, \tag{2.3b}$$

where

$$G = (3/4)(M/E)(a/r_M^3)\,, \tag{2.3c}$$

with M and E being the moon's and the earth's masses, respectively, a being the mean radius of the earth (6,371 km) and r_M being the mean earth-moon distance; according to the values listed in Table 1 of [US], $G \simeq 0.41865 \times 10^{-7}$. Further,

$$G_a(\phi) = (1 - 3\sin^2\phi)\,, \tag{2.4a}$$

$$G_b(\phi) = \sin 2\phi\,, \tag{2.4b}$$

$$G_c(\phi) = \cos^2\phi\,. \tag{2.4c}$$

In [Blaha, 1981], Table 1 was constructed featuring the constituent heights expressed according to (2.1) - (2.4c). Its "extreme magnitude" column features the largest values of h_j which can be reached as follows:

a ... $\phi = \pm 90^0$,
b ... $\phi = \pm 45^0$,
c ... $\phi = 0$.

In each group (a, b, or c) the constituents are listed in the descending order of magnitude. This table is reproduced here for the sake of completeness.

KIND	SYMBOL	DESCRIPTION	C_j	h_j	EXTREME MAGNITUDE
a	A_o	constant	.7384	$.0985\text{m}\ (1-3\sin^2\phi)$	.197m
	Mf	semimonthly	.1566	$.0209\text{m}\ (1-3\sin^2\phi)\cos\alpha_{Mf}$	.042m
	Mm	monthly	.0827	$.0110\text{m}\ (1-3\sin^2\phi)\cos\alpha_{Mm}$	.022m
	SSa	semiannual	.0728	$.0097\text{m}\ (1-3\sin^2\phi)\cos\alpha_{SSa}$	.019m
b	K_1	declinational luni-solar	.5305	$.1415\text{m}\ \sin 2\phi\ \cos\alpha_{K_1}$	.141m
	O_1	principal lunar	.3771	$.1006\text{m}\ \sin 2\phi\ \cos\alpha_{O_1}$	.101m
	P_1	principal solar	.1755	$.0468\text{m}\ \sin 2\phi\ \cos\alpha_{P_1}$	.047m
c	M_2	principal lunar	.9085	$.2423\text{m}\ \cos^2\phi\ \cos\alpha_{M_2}$	.242m
	S_2	principal solar	.4227	$.1127\text{m}\ \cos^2\phi\ \cos\alpha_{S_2}$	.113m
	N_2	ecliptical lunar	.1759	$.0469\text{m}\ \cos^2\phi\ \cos\alpha_{N_2}$	.047m
	K_2	declinational luni-solar	.1151	$.0307\text{m}\ \cos^2\phi\ \cos\alpha_{K_2}$	.031m

Table 1

Approximate heights of selected tidal constituents, including extreme magnitudes

2.1 Long-Period Constituents

If only the average effects were of interest, the constituent heights h_j in the group "a" of Table 1 could be compared with the formulas of [Lisitzin, 1974], pages 49 and 38, 39, respectively, implying for the tidal potential of the long-period terms:

$$W_{A_o} = 0.96621 \ (1\text{-}3 \sin^2\phi) \ m^2/sec^2,$$

$$W_{Mf} = 0.20460 \ (1\text{-}3 \sin^2\phi) \cos 2s \ m^2/sec^2,$$

$$W_{Mm} = 0.10796 \ (1\text{-}3 \sin^2\phi) \cos(s\text{-}p) \ m^2/sec^2,$$

$$W_{SSa} = 0.09531 \ (1\text{-}3 \sin^2\phi) \cos 2h \ m^2/sec^2,$$

where

h = mean longitude of the sun,
s = mean longitude of the moon,
p = longitude of the lunar perigee.

It follows for the constituent heights:

$$W_{A_o} / \bar{g} \simeq 0.0986m \ (1\text{-}3 \sin^2\phi) \ ,$$

$$W_{Mf} / \bar{g} \simeq 0.0209m \ (1\text{-}3 \sin^2\phi) \cos 2s \ ,$$

$$W_{Mm} / \bar{g} \simeq 0.0110m \ (1\text{-}3 \sin^2\phi) \cos(s\text{-}p) \ ,$$

$$W_{SSa} / \bar{g} \simeq 0.0097m \ (1\text{-}3 \sin^2\phi) \cos 2h \ ,$$

where $\bar{g}$, the average terrestrial gravity, is adopted as 9.80 m/sec^2. Except for the less explicit notations for the arguments "α" in Table 1, the agreement is nearly perfect.

If a more accurate representation is sought, the influence of the moon and the sun must be treated separately. For this purpose, the node factor for the moon, denoted in general as f_j, can no longer be assumed to be unity (it is always unity for the sun). When applied separately for the moon and the sun, (2.2) becomes

$$A_j = K_j \, G_j(\phi) \, C_j \, f_j \; \ldots \text{ moon }, \tag{2.5a}$$

$$A'_j = K_j \, G_j(\phi) \, C'_j \quad \ldots \text{ sun }. \tag{2.5b}$$

The node factor (a function of the longitude of the moon's node with the periodicity of about 18.6 years) changes very slowly from year to year for each constituent. Table 14 of [US] gives the value of the pertinent f_j for the middle of each year between 1850 and 1999.

If a given constituent represents the moon's action alone, (2.1) is adopted without change in notations and A_j is computed according to (2.5a). For a strictly solar constituent, A_j and α_j in (2.1) are replaced by A'_j and α'_j, with A'_j computed as in (2.5b). If a constituent is composed of both effects, the resulting h_j is obtained as

$$h_j = A_j \cos\alpha_j + A'_j \cos\alpha'_j = K_j G_j(\phi)(C_j f_j \cos\alpha_j + C'_j \cos\alpha'_j). \tag{2.6}$$

For the permanent tide, the value which corresponds to the active life-span of SEASAT can be associated with mid-year 1978 and is given as

$$f_{A_0} = f_{Mm} = 1.131 \, . \tag{2.7}$$

Since α_j and α'_j are immaterial for the permanent tide, with the aid of (2.3a) and (2.4a) equation (2.6) becomes

$$h_{A_0} = 0.13335\text{m}\ (1-3\sin^2\phi)(C_{A_0} f_{A_0} + C'_{A_0}) \ . \tag{2.8}$$

Table 2 of [US] gives

$$C_{A_0} = 0.5044, \qquad C'_{A_0} = 0.2340 \ ,$$

which, when added algebraically, yield the value 0.7384 seen in Table 1 for an average effect. However, in considering the specific case of SEASAT altimetry and the corresponding value f_{A_0} in (2.7), equation (2.8) yields the explicit form for the equilibrium height of the constituent A_0:

$$h_{A_0} = 0.1073\text{m}\ (1-3\sin^2\phi) \ . \tag{2.9}$$

The constituents Mf and Mm are due exclusively to the moon. In agreement with [US] the tidal arguments are written as

$$\alpha_{Mf} = \alpha'_{Mf} - 2\xi \ ,$$
$$\alpha_{Mm} = \alpha'_{Mm} \ ;$$

Table 6 or Table 11 of [US] give for the SEASAT observational epoch:

$$-2\xi = -1.07^0 \ .$$

The coefficients "C" and the node factors "f" for this epoch are

$$C_{Mf} = 0.1566 \ , \qquad f_{Mf} = 0.625 \ ,$$
$$C_{Mm} = 0.0827 \ , \qquad f_{Mm} = 1.131 \ .$$

According to (2.1) and (2.5a), the equilibrium formulas representing the SEASAT epoch thus read

$$h_{Mf} = 0.0131\mathrm{m}\ (1-3\ \sin^2\phi)\ \cos\ (\alpha'_{Mf} - 1.07^\circ)\ , \tag{2.10}$$

$$h_{Mm} = 0.0125\mathrm{m}\ (1-3\ \sin^2\phi)\ \cos\alpha'_{Mm}\ . \tag{2.11}$$

The constituent SSa owes its existence to the sun only, hence (2.1) and (2.5b) apply. According to [US], one can write

$$\alpha_{SSa} = \alpha'_{SSa}\ ,$$

$$C_{SSa} = 0.0728\ ,$$

yielding, regardless of the SEASAT epoch,

$$h_{SSa} = 0.0097\mathrm{m}\ (1-3\ \sin^2\phi)\ \cos\alpha'_{SSa}\ , \tag{2.12}$$

which is essentially (except for the argument notation) the same as the corresponding expression in Table 1.

2.2 Diurnal Constituents

When considering the constituent K_1, one realizes that the value h_{K_1} is made up of the moon's and the sun's contribution, hence it is of the form (2.6). Due to the changing longitude (N) of the moon's node, α_{K_1} differs from α'_{K_1} by a small quantity $-\nu$, namely

$$\alpha_{K_1} = \alpha'_{K_1} - \nu . \qquad (2.13)$$

Similar to the node factor, the periodicity of ν is approximately 18.6 years. Over short periods (e.g. less than a year) it can be considered constant. Table 6 of [US] gives the values of ν according to N. For the beginning of September 1978, the epoch which is quite representative of the SEASAT data series, the value of N found from Table 4 of [US] is

$$N \simeq 178^0 , \qquad (2.14)$$

implying that

$$\nu \simeq 0.57^0 . \qquad (2.15)$$

The value in (2.14) is considered constant for all SEASAT observations.

In order to obtain h_{K_1} from (2.6) in the form similar to (2.1) in conjunction with (2.5a), one has to simplify the following expression:

$$c_{K_1} = C_{K_1} f_{K_1} \cos(\alpha'_{K_1} - \nu) + C'_{K_1} \cos\alpha'_{K_1} , \qquad (2.16a)$$

which corresponds to the quantity inside the parentheses in (2.6) with K_1 substituted for j and with (2.13) taken into account. Equation (2.16a) can be developed into

$$c_{K_1} = (C_{K_1} f_{K_1} \cos\nu + C'_{K_1}) \cos\alpha'_{K_1} + C_{K_1} f_{K_1} \sin\nu \sin\alpha'_{K_1} , \qquad (2.16b)$$

which is of the form

$$C = C_1 \cos\alpha + C_2 \sin\alpha = (C_1^2 + C_2^2)^{\frac{1}{2}} [(C_1 \cos\alpha + C_2 \sin\alpha)/(C_1^2 + C_2^2)^{\frac{1}{2}}].$$

If ν' is defined as arc tg (C_2/C_1), one has

$$\sin\nu' = C_2/(C_1^2 + C_2^2)^{\frac{1}{2}} , \qquad \cos\nu' = C_1/(C_1^2 + C_2^2)^{\frac{1}{2}} ,$$

and hence

$$C = (C_1^2 + C_2^2)^{\frac{1}{2}} \cos (\alpha - \nu') .$$

When applied to (2.16b) this yields

$$c_{K_1} = (\tilde{C}_{K_1} \tilde{f}_{K_1}) \cos\tilde{\alpha}_{K_1} , \qquad (2.17a)$$

where

$$\tilde{C}_{K_1} \tilde{f}_{K_1} = (C_1^2 + C_2^2)^{\frac{1}{2}} , \qquad (2.17b)$$

$$C_1 = C_{K_1} f_{K_1} \cos\nu + C'_{K_1} , \qquad C_2 = C_{K_1} f_{K_1} \sin\nu , \qquad (2.17c)$$

$$\tilde{\alpha}_{K_1} = \alpha'_{K_1} - \nu' , \qquad \nu' = \text{arc tg } (C_2/C_1) . \qquad (2.17d)$$

The coefficient $\tilde{C}_{K_1}$ represents a certain mean value defined as

$$\tilde{C}_{K_1} = \text{mean} [(C_1^2 + C_2^2)^{\frac{1}{2}} \cos\nu'] \equiv \text{mean } C_1 , \qquad (2.18a)$$

which in Table 2 of [US] is listed to be

$$\tilde{C}_{K_1} = 0.5305 \ . \tag{2.18b}$$

Further listed are the values

$$C_{K_1} = 0.3623 \ , \qquad C'_{K_1} = 0.1681 \ .$$

The node factor for K_1 is identical to that for J_1 of equation (76) of [US], and it is listed for mid-1978 as

$$f_{K_1} = f_{J_1} = 0.827.$$

With these values for C_{K_1}, C'_{K_1}, f_{K_1}, and with ν from (2.15), one calculates (see 2.17 c,d):

$$C_1 \simeq 0.4677, \qquad C_2 \simeq 0.002981, \qquad \nu' \simeq 0.37 \ ;$$

from here it follows (see 2.17 b,d) that

$$\tilde{C}_{K_1}\tilde{f}_{K_1} = 0.4677 \ , \qquad \tilde{\alpha}_{K_1} = \alpha'_{K_1} - 0.37^{\circ}. \tag{2.19}$$

The values in (2.19) could also be obtained more directly from Table 14, and Table 6 or Table 11 of [US], respectively. In particular, 0.4677 could be found upon multiplying 0.5305 in (2.18b) by the corresponding node factor $\tilde{f}_{K_1}$, listed for mid-1978 in Table 14 (under the heading K_1) as 0.882; and 0.37° could be found, for $N = 178^{\circ}$, from Table 6 (under the heading ν') or from Table 11 (under the heading K_1). In either case the equilibrium formula (2.6), applied to K_1 in conjunction with (2.3b), (2.4b), (2.17a) and (2.19), becomes

$$h_{K_1} = 0.1248m \sin 2\phi \cos(\alpha'_{K_1} - 0.37^{\circ}) \ , \tag{2.20}$$

referring to the epoch of SEASAT altimeter data acquisition. The variable part of the argument for this and other tidal constituents, α_j', will be described later.

The effect of O_1 is due exclusively to the moon, hence (2.1) and (2.5a) apply. In analogy to (2.13) and the development that followed, α_{O_1} will be written as α_{O_1}' (this, in itself, is immaterial here) plus some small quantity which will again be considered constant due to the short active life-span of SEASAT. In particular,

$$\alpha_{O_1} = \alpha_{O_1}' + (2\xi - \nu) . \qquad (2.21)$$

For N given in (2.14), Table 4 of [US] yields approximately 0.54° for ξ and 0.57° for ν (see equation 2.15 above); Table 11 of the same reference yields directly $2\xi - \nu$ under the heading O_1. *In either case the result is*

$$2\xi - \nu = 0.50^\circ . \qquad (2.22)$$

The coefficient "C" and the node factor for mid-1978 are

$$C_{O_1} = 0.3771, \qquad f_{O_1} = 0.806. \qquad (2.23)$$

Upon inserting the results (2.3b), (2.4b) and (2.21) - (2.23) into (2.1) and (2.5a), one has the equilibrium formula for O_1 representing the SEASAT observational epoch:

$$h_{O_1} = 0.0811\text{m} \sin 2\phi \cos(\alpha_{O_1}' + 0.50^\circ) . \qquad (2.24)$$

The constituent P_1 is due to the sun's effect. From [US] we extract

$$\alpha_{P_1} = \alpha'_{P_1} ,$$

$$C_{P_1} = 0.1755 ,$$

yielding

$$h_{P_1} = 0.0468\text{m} \sin 2\phi \cos\alpha'_{P_1} . \qquad (2.25)$$

This expression is independent of the SEASAT epoch and is essentially identical to the corresponding expression in Table 1.

2.3 Semidiurnal Constituents

The most important of all the tidal constituents, M_2, is due exclusively to the moon. It can be developed in a complete analogy to the approach followed for O_1. The argument is written as

$$\alpha_{M_2} = \alpha'_{M_2} + (2\xi - 2\nu) , \tag{2.26}$$

where ξ and ν were already found; thus

$$2\xi - 2\nu = -0.07^{\circ} , \tag{2.27}$$

which is also the value given in Table 11 of [US] under the heading M_2. Further, one has

$$C_{M_2} = 0.9085 , \qquad f_{M_2} = 1.038 , \tag{2.28}$$

and

$$h_{M_2} = 0.2515\text{m} \cos^2\phi \cos(\alpha'_{M_2} - 0.07^{\circ}) , \tag{2.29}$$

which is the equilibrium formula for M_2 corresponding to the SEASAT observational epoch. It has been obtained from (2.26) - (2.28) in the same way as (2.24) was obtained from (2.21) - (2.23) except, of course, that (2.4b) has been replaced by (2.4c).

The constituent S_2 owes its existence to the sun. The argument is thus written as α'_{S_2} in agreement with the original convention, and the node factor is omitted. In other respects the equilibrium formula for S_2 is derived similar to (2.29) above, namely

$$h_{S_2} = 0.1127\text{m} \cos^2\phi \cos\alpha'_{S_2} . \tag{2.30}$$

In this case, no special considerations related to the SEASAT observational epoch are necessary.

The constituent N_2 is due solely to the moon; one has

$$\alpha_{N_2} = \alpha'_{N_2} + (2\xi - 2\nu) ,$$

where, according to (2.27),

$$2\xi - 2\nu = -0.07^0 .$$

Furthermore, one finds

$$C_{N_2} = 0.1759, \qquad f_{N_2} = 1.038 ,$$

hence

$$h_{N_2} = 0.0487\text{m} \cos^2\phi \cos(\alpha'_{N_2} - 0.07^0) , \tag{2.31}$$

which is the equilibrium formula for N_2 corresponding to the SEASAT epoch.

The constituent K_2 reflects the effect of both the moon and the sun, hence (2.6) applies, where

$$\alpha_{K_2} = \alpha'_{K_2} - 2\nu .$$

Accordingly,

$$h_{K_2} = Ga \cos^2\phi \, c_{K_2} ,$$

where Ga was given in (2.3b) and where

$$c_{K_2} = C_{K_2} f_{K_2} \cos(\alpha'_{K_2} - 2\nu) + C'_{K_2} \cos \alpha'_{K_2} ,$$

in analogy to (2.16a). The numerical values are

$$2\nu = 1.14^0 ;$$

$$C_{K_2} = 0.0786 , \qquad f_{K_2} = 0.630 ;$$

$$C'_{K_2} = 0.0365 .$$

From this point on, the algorithm contained in the formulas (2-16a)-(2.19) can be adopted in its entirety, except for the following changes in notations:

$$K_1 \to K_2 , \qquad \nu \to 2\nu , \qquad \nu' \to 2\nu'' .$$

The results are

$$c_{K_2} = (\tilde{C}_{K_2}\tilde{f}_{K_2}) \cos\tilde{\alpha}_{K_2} ;$$

$$\tilde{C}_{K_2}\tilde{f}_{K_2} = 0.08602 ,$$

$$\tilde{\alpha}_{K_2} = \alpha'_{K_2} - 2\nu'' ,$$

$$2\nu'' = 0.66^0 .$$

Thus

$$h_{K_2} = 0.0230\text{m} \cos^2\phi \cos(\alpha'_{K_2} - 0.66^0) . \qquad (2.32)$$

This outcome could also be obtained from Tables 2, 14, and 6 or 11 of [US]. In particular, Table 2 lists $\tilde{C}_{K_2}$ (see Note 4 on page 166) as 0.1151. The corresponding node factor $\tilde{f}_{K_2}$ is 0.748 appearing, for the mid-1978, in Table 14; this yields the product 0.08609 which is in a good agreement with the above computed value. Table 6 gives $2\nu''$ for $N = 178^0$ (see equation 2.14) as 0.66^0; equivalently, Table 11 shows this value under the heading K_2.

2.4 Tidal Arguments

The explicit expressions for the equilibrium tidal arguments are developed in a way similar to Table 2 of [US] with a few minor changes. One change pertains to the constant part "u" which is presently expressed numerically (in ^{0}) and represents the SEASAT observational epoch. With regard to the computation of α_j', the variable part of the argument (in Table 2 of [US] denoted as V), UT ±12 hours is used instead of T, the hour angle of the mean sun at Greenwich at the time of the tidal evaluation. Since all the quantities will be considered as given in degrees instead of hours, the following applies:

$$T \;=\; UT \pm 180^0 \;, \tag{2.33}$$

where UT (in ^{0}) is obtained by multiplying UT (in hours) by 15 (^{0}/hour), etc. The other two variables needed for the evaluation of α_j' at Greenwich for the desired constituents are h, the mean longitude of the sun, s, the mean longitude of the moon and p, the longitude of the lunar perigee. In terms of local -- rather than Greenwich -- arguments, UT is replaced by $UT+\lambda$, where λ (in ^{0}) is the customary east longitude of the point where the tidal evaluation is sought, symbolized by

$$\text{local argument} \ldots \; UT \rightarrow UT + \lambda \;. \tag{2.34}$$

In order to indicate the computation of the equilibrium tidal arguments at Greenwich, Table 2 lists these arguments in two parts (see its second column), α_j' and u_j; the final argument is

$$\alpha_j \;=\; \alpha_j' + u_j \;. \tag{2.35}$$

CONSTITUENT	GREENWICH ARGUMENT α'_j	$+u_j$	SPEED °/day	°/hour	α'_j Jan 0.5, 1900	PERIOD
Mf	2s	-1.07°	26.352793539	1.09803306	180.874844°	13.6608 d
Mm	s - p	+0.00	13.064992739	0.54437470	296.109403°	27.5546 d
SSa	2h	+0.00	1.971294671	0.08213728	199.393356°	182.6211 d
K_1	UT + h + 90°	-0.37°	360.985647335	15.04106864	189.696678°	23.9345 hr
O_1	UT - 2s + h - 90°	+0.50°	334.632853797	13.94303557	188.821833°	25.8193 hr
P_1	UT - h - 90°	+0.00	359.014352665	14.95893136	170.303322°	24.0659 hr
M_2	2UT - 2s + 2h	-0.07°	695.618501132	28.98410421	18.518511°	12.4206 hr
S_2	2UT	+0.00	720.	30.	0.	12 hr
N_2	2UT - 3s + 2h + p	-0.07°	682.553508393	28.43972952	82.409108°	12.6583 hr
K_2	2UT + 2h	-0.66°	721.971294671	30.08213728	199.393356°	11.9672 hr

Table 2

Greenwich arguments and related quantities for selected equilibrium tidal constituents

The α'_j part agrees with [Schwiderski, 1980], page 172, and with [Lisitzin, 1974], page 12.

For the explicit computation of α'_j, the expressions for h, s and p are adapted from [US], page 162, as

$$h = 279.696678^0 + 36{,}000.768925^0 T + 0.000303^0 T^2 \,, \tag{2.36a}$$

$$s = 270.437422^0 + 481{,}267.892000^0 T + 0.002525^0 T^2 + 0.000002 T^3 \,, \tag{2.36b}$$

$$p = 334.328019^0 + 4{,}069.032206^0 T - 0.010344^0 T^2 - 0.000012^0 T^3 \,, \tag{2.36c}$$

where T is the number of Julian centuries (of 36,525 days) reckoned from January 0.5, 1900 at Greenwich, i.e., from December 31, 1899, 12h UT. For January 0.0, 1978 at Greenwich, the value of T is 28,488.5/36,525; upon considering (2.36 a-c) one has

$$[h] = 279.310976^0 , \quad [s] = 166.218322^0 , \quad [p] = 268.055437^0 , \tag{2.37}$$

where the brackets have been used to indicate this specific time epoch. Near a point of expansion, i.e., certainly within a year, h, s and p can be considered as linear functions of time and their speeds in ^{0}/day, etc., can be evaluated using the terms linear in T in (2.36 a-c). When considered together with (2.37), these speeds make it possible to compute h, s and p for any instant in 1978 accurately as

$$\begin{aligned} h = 279.310976^0 &+ 0.985647335^0 \cdot D + 0.04106864^0 \cdot hr \\ &+ 0.00068448^0 \cdot min + 0.00001141^0 \cdot sec \,, \end{aligned} \tag{2.38a}$$

$$\begin{aligned} s = 166.218322^0 &+ 13.176396769^0 \cdot D + 0.54901653^0 \cdot hr \\ &+ 0.00915028^0 \cdot min + 0.00015250^0 \cdot sec \,, \end{aligned} \tag{2.38b}$$

$$\begin{aligned} p = 268.055437^0 &+ 0.111404030^0 \cdot D + 0.00464183^0 \cdot hr \\ &+ 0.00007736^0 \cdot min + 0.00000129^0 \cdot sec \,, \end{aligned} \tag{2.38c}$$

where D = day number in 1978, and hr, min, sec represent hours, minutes, seconds in UT for that day. From the formulas (2.38 a-c) the various rates of change in h, s and p are apparent. They also confirm the periodicity of h (365.2421988 days = tropical year), of s (27.32158164 days = tropical month) and of p (8.847313 years of 365.25 days).

The numerical values of α'_j at Greenwich for any instant in 1978 can be found from the general expression appearing in the second column of Table 2. The rate of change in UT, taken in the interval 0-24 hours, is 15^0/hr, 0.25^0/min and 0.00416667^0/sec, while the initial values and the rates of change in the other three variables, h, s and p, have been given in (2.38 a-c). The required combinations of UT, h, s and p thus yield

$$\alpha'_{Mf} = 332.436645^0 + 26.352793539^0 * D + 1.09803306^0 * hr + 0.01830055^0 * min + 0.00030501^0 * sec \ , \tag{2.39a}$$

$$\alpha'_{Mm} = 258.162886^0 + 13.064992739^0 * D + 0.54437470^0 * hr + 0.00907291^0 * min + 0.00015122^0 * sec \ , \tag{2.39b}$$

$$\alpha'_{SSa} = 198.621952^0 + 1.971294671^0 * D + 0.08213728^0 * hr + 0.00136895^0 * min + 0.00002282^0 * sec \ ; \tag{2.39c}$$

$$\alpha'_{K_1} = 9.310976^0 + 0.985647335^0 * D + 15.04106864^0 * hr + 0.25068448^0 * min + 0.00417807^0 * sec \ , \tag{2.40a}$$

$$\alpha'_{O_1} = 216.874331^0 - 25.367146203^0 * D + 13.94303557^0 * hr + 0.23238393^0 * min + 0.00387307^0 * sec \ , \tag{2.40b}$$

$$\alpha'_{P_1} = 350.689024^0 - 0.985647335^0 * D + 14.95893136^0 * hr + 0.24931552^0 * min + 0.00415526^0 * sec \ ; \tag{2.40c}$$

$$\alpha'_{M_2} = 226.185307^0 - 24.381498868^0 * D + 28.98410421^0 * hr + 0.48306840^0 * min + 0.00805114^0 * sec \ , \tag{2.41a}$$

$$\alpha'_{S_2} = 30.^0 * hr + 0.5^0 * min + 0.00833333^0 * sec \ , \tag{2.41b}$$

$$\alpha'_{N_2} = 328.022421^0 - 37.446491607^0 * D + 28.43972952^0 * hr + 0.47399549^0 * min + 0.00789992^0 * sec , \qquad (2.41c)$$

$$\alpha'_{K_2} = 198.621952^0 + 1.971294671^0 * D + 30.08213728^0 * hr + 0.50136895^0 * min + 0.00835615^0 * sec , \qquad (2.41d)$$

where D, hr, min, sec were defined following (2.38 a-c). From these formulas the rates of change in the arguments α'_j and thus also α_j are apparent and agree with Table 2 of [US] wherever they are comparable (i.e., they agree with the values printed in [US] as "speed per solar hour" which, however, exhibit fewer significant digits than the speeds derived above). Furthermore, these rates also agree with [Estes, 1980], page 118, and with [Godin, 1972], page 232; they agree approximately with [Schwiderski, 1980], page 172, [Estes, 1980], page 101, and [Lisitzin, 1974], page 12. The rates associated with "D" and "hr" are further presented in Table 2, columns 3 and 4, respectively, under the headings ^{0}/day and ^{0}/hour.

The fifth column of Table 2 lists α'_j at Greenwich for January 0.5, 1900 obtained, with the aid of the second column, from (2.36 a-c) for T=0. One could evaluate α'_j at any instant also from these values upon applying the rates listed in the columns 3 and 4. However, this would lead to a slight loss of accuracy even if sufficient digits are used in the arithmetic, due to neglecting the terms in T^2 (and T^3) inherent in the formulas for h, s and p in (2.36 a-c). By comparison, the terms in T^2 and T^3 did enter (2.38 a-c) and thus also the expressions that followed, developed herein in view of SEASAT altimetry. The latter formulas are advantageous to use not only for their accuracy, but also because they are very simple and do not necessitate a large number of significant digits for their evaluation.

Having found all of the α'_j , one can recapitulate, with the aid of Table 2, Greenwich and local arguments corresponding to SEASAT observational epoch. For the long-period tides there is no distinction between these two kinds and we have

$$\alpha_{A_0} = 0 , \tag{2.42a}$$

$$\alpha_{Mf} = \alpha'_{Mf} - 1.07^{\circ} , \tag{2.42b}$$

$$\alpha_{Mm} = \alpha'_{Mm} , \tag{2.42c}$$

$$\alpha_{SSa} = \alpha'_{SSa} . \tag{2.42d}$$

Greenwich arguments for the diurnal constituents similarly are

$$\alpha_{K_1} = \alpha'_{K_1} - 0.37^{\circ} , \tag{2.43a}$$

$$\alpha_{O_1} = \alpha'_{O_1} + 0.50^{\circ} , \tag{2.43b}$$

$$\alpha_{P_1} = \alpha'_{P_1} , \tag{2.43c}$$

while Greenwich arguments for the semidiurnal constituents are

$$\alpha_{M_2} = \alpha'_{M_2} - 0.07^{\circ} , \tag{2.44a}$$

$$\alpha_{S_2} = \alpha'_{S_2} , \tag{2.44b}$$

$$\alpha_{N_2} = \alpha'_{N_2} - 0.07^{\circ} , \tag{2.44c}$$

$$\alpha_{K_2} = \alpha'_{K_2} - 0.66^{\circ} . \tag{2.44d}$$

The above three groups of formulas appeared, in a similar form, in Sections 2.1, 2.2 and 2.3, respectively.

The local arguments (for the diurnal and semidiurnal constituents) then read

$$\tilde{\alpha}_{K_1} = \alpha'_{K_1} + \lambda - 0.37^0 , \tag{2.45a}$$

$$\tilde{\alpha}_{O_1} = \alpha'_{O_1} + \lambda + 0.50^0 , \tag{2.45b}$$

$$\tilde{\alpha}_{P_1} = \alpha'_{P_1} + \lambda ; \tag{2.45c}$$

$$\tilde{\alpha}_{M_2} = \alpha'_{M_2} + 2\lambda - 0.07^0 , \tag{2.45d}$$

$$\tilde{\alpha}_{S_2} = \alpha'_{S_2} + 2\lambda , \tag{2.45e}$$

$$\tilde{\alpha}_{N_2} = \alpha'_{N_2} + 2\lambda - 0.07^0 , \tag{2.45f}$$

$$\tilde{\alpha}_{K_2} = \alpha'_{K_2} + 2\lambda - 0.66^0 . \tag{2.45g}$$

3. TIDAL MATHEMATICAL MODEL

3.1 Simplified Tidal Model

This model encompasses the equilibrium tide in conjunction with the solid earth deformation as described on pages 31 and 32 of [Blaha, 1981]. In that report, abbreviated here as [B], the water level was assumed to conform instantaneously to the total tidal potential $\tilde{\Gamma}$. This potential, in turn, was assumed to be

$$\tilde{\Gamma} = (1+k)\Gamma \ , \tag{3.1}$$

where Γ is the equilibrium, or astronomical, tidal potential (due to the effect of the moon and the sun) and k is one of the Love numbers; $k\Gamma$ is then the "additional tidal potential" owing its existence to the earth deformation. Equation (3.1) was not written explicitly per se, but it was implied in the form of the "geocentric tide" listed below.

The simplifications present in the model of [B] discarded the forces of friction and viscosity as well as various other effects; of these, the self-gravitation and the ocean loading effects will be discussed in the first part of the next section. The heights of the various tidal phenomena as described in [B] are now briefly recapitulated:

$$\text{equilibrium tide} = \Gamma/g \ , \tag{3.2}$$

$$\text{earth deformation} = h\Gamma/g \ , \tag{3.3}$$

$$\text{additional tide (due to the earth deformation)} = k\Gamma/g \ ; \tag{3.4}$$

$$\text{"geocentric tide"} = (1+k)\Gamma/g \ , \tag{3.5}$$

$$\text{"measured tide"} = (1+k-h)\Gamma/g \ ; \tag{3.6}$$

$$\text{"geocentric tide"} = [(1+k)/(1+k-h)] \ \text{"measured tide"} \ . \tag{3.7}$$

The symbol g in these formulas represents the average value of gravity. Equation (3.7) is the consequence of (3.5) and (3.6). In [B], the values of the Love numbers h, k were adopted from [Vaníček, 1980], pages 10 and 11, as

$$h \simeq 0.62 \ , \qquad k \simeq 0.29 \ . \tag{3.8}$$

3.2 Realistic Tidal Model

In this model, the above mentioned forces of friction and viscosity as well as the self-gravitation and the ocean loading effects will be accounted for in the Laplace Tidal Equations which include also the Coriolis force and the spatial distribution of depth. The development will begin with the self-gravitation to which the ocean loading effects (the effect on the ocean bottom, the effect on the potential) will be related subsequently. The Laplace Tidal Equations (LTE) will be recapitulated using a fair amount of detail, which will shed light on various possible simplifications and, especially, on the surface tide directly related to altimeter observations. Finally, a model expressing the surface tide for the diurnal and semidiurnal tidal constituents of interest will be derived. The following four references will be mentioned on numerous occasions: [Vaníček, 1980], [Estes, 1980], [Parke and Hendershott, 1980] and [Schwiderski, 1980]; they will be abbreviated in the text as [V], [E], [PH] and [S], respectively.

Effect of the self-gravitation of tidal waters on the potential. We begin by presenting the effect of (discontinuous) point masses on the potential at point i, denoted V_s:

$$V_s = G \sum_j M_j/\ell_{ij} ,$$

where

G = gravitational constant,

M_j = mass of the j-th point mass,

ℓ_{ij} = distance between the point i (of evaluation) and the j-th point mass.

If M_j is replaced by $\rho_j \xi_j d\Omega_j$, where ρ is the average density of mass (here water) in the column of height ξ and cross-section $d\Omega$, the potential becomes

$$V_s = G \sum_j \rho_j \xi_j \ell_{ij}^{-1} \, d\Omega_j \ . \tag{3.9a}$$

Since water has continuous distribution we can write

$$V_s = G \iint_S \rho\xi \, |\overline{x} - \overline{x}'|^{-1} \, dS' \ , \tag{3.9b}$$

where, in addition to the symbols already introduced (e.g., ξ represents the height of the tidal waters), the following representations apply:

$\overline{x}$ ... position where V_s is being evaluated ,

$\overline{x}'$ ... position of the element of integration ,

dS'... surface element ,

S ... integration domain, the ocean surface.

Equation (3.9b) is essentially (55) of [V] (the signs in this reference indicate "correction" rather than "effect").

The simplifications to be used throughout are the spherical approximation and the assumption of constant density of the sea water. The following relations thus hold true only approximately, but the sign of equality can nevertheless be used if one keeps in mind this limitation. Under these circumstances, from (3.9b) we obtain

$$V_s = GR^2\rho \iint_\Omega \xi \, |x - x'|^{-1} \, d\Omega' \ , \tag{3.10}$$

where

R = mean radius of the earth ,

ρ = mean density of the sea water ,

$d\Omega'$ = solid angle element (e.g., $\cos\phi' d\phi' d\lambda'$) ,

Ω = ocean surface over the unit sphere .

Equation (3.10) is, in fact, the expression for V_s on page 105 of [E]. All the relations written in terms of G and R could also be written in terms of g according to

$$g = GM_e/R^2, \tag{3.11a}$$

or

$$G\rho R = gR^3\rho/M_e , \tag{3.11b}$$

where M_e is the earth's mass.

The distance $\overline{x}$, $\overline{x}'$ can be expressed as

$$|\overline{x} - \overline{x}'| = R\,|\tilde{x} - \tilde{x}'| , \tag{3.12}$$

where the points $\tilde{x}$, $\tilde{x}'$ are now located on a unit sphere; either pair is separated by the central angle ψ such that

$$\cos\psi = \sin\phi \sin\phi' + \cos\phi \cos\phi' \cos(\lambda' - \lambda) , \tag{3.13}$$

where ϕ,λ are the latitude and the longitude, respectively. The reciprocal value of the distance $\tilde{x}$, $\tilde{x}'$ may be expanded in an infinite series of Legendre polynomials in the argument $\cos\psi$, $P_n(\cos\psi)$, so that we can write

$$|\tilde{x} - \tilde{x}'|^{-1} = (2 \sin \tfrac{1}{2}\psi)^{-1} = [2(1 - \cos\psi)]^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(\cos\psi) . \tag{3.14}$$

With (3.12) and (3.14), equation (3.10) becomes

$$V_s = (gR^3\rho/M_e)\iint_{\Omega} \xi(2\sin\tfrac{1}{2}\psi)^{-1}\, d\Omega' = (gR^3\rho/M_e)\iint_{\Omega} \xi\,[2(1-\cos\psi)]^{-\frac{1}{2}}\, d\Omega'$$

$$= (gR^3\rho/M_e)\iint_{\Omega} \xi \sum_n P_n(\cos\psi)\, d\Omega' \ , \qquad (3.15)$$

where (3.11b) has also been taken into account and where the notation

$$\sum_n \equiv \sum_{n=0}^{\infty}$$

has been introduced. The first expression in (3.15) corresponds to a formula on page 106 and to one building block of (A10) in [E], the second expression corresponds to (59) of [V], and the third expression corresponds to (56) of [V].

Because the integrand in the last expression of (3.15) is a well-behaved function, the summation can be interchanged with the integration and we have

$$V_s = (gR^3\rho/M_e) \sum_n \iint_{\Omega} \xi\, P_n(\cos\psi)\, d\Omega' = \sum_n (V_s)_n \ , \qquad (3.16a)$$

$$(V_s)_n = (gR^3\rho/M_e)\iint_{\Omega} \xi\, P_n(\cos\psi)\, d\Omega' \ . \qquad (3.16b)$$

In these equations the potential V_s is expanded in terms of surface spherical harmonics (this is always possible for a function on a sphere) and its explicit form is given.

The vertical displacement of an equipotential surface caused by the self-gravitation is

$$u_a = V_s/g \ , \qquad (3.17)$$

which is the same as equation (60) in [V]. Upon dividing (3.15) by g, one obtains

$$u_a = (R^3\rho/M_e)\iint_\Omega \xi\,(2\sin\tfrac{1}{2}\psi)^{-1}\, d\Omega' \ , \tag{3.17'}$$

which could be further expanded to include the other two expressions in (3.15). From (3.16a,b) it similarly follows that

$$u_a = (R^3\rho/M_e)\sum_n \iint_\Omega \xi\, P_n(\cos\psi)\, d\Omega' = \sum_n (u_a)_n \ , \tag{3.18a}$$

$$(u_a)_n = (V_s)_n/g = (R^3\rho/M_e)\iint_\Omega \xi\, P_n(\cos\psi)\, d\Omega' \ . \tag{3.18b}$$

The formula giving $(u_a)_n$ can be rewritten if one takes advantage of the result from Appendix 2 (see A2.6) derived in the spherical coordinates (θ,λ), where θ is the colatitude. If ρ_e further denotes the average earth density,

$$\rho_e = M_e/[(4/3)\pi R^3] \ ,$$

equation (3.18b) becomes

$$(u_a)_n = [3\rho/(4\pi\rho_e)]\iint_\Omega \xi(\theta',\lambda')\, P_n(\cos\psi)\, d\Omega' \ .$$

But according to Appendix 2 the last integral is $[4\pi/(2n+1)]\xi_n(\theta,\lambda)$, hence

$$(u_a)_n = [3/(2n+1)]\,(\rho/\rho_e)\,\xi_n \ ,$$

where $\xi_n \equiv \xi_n(\theta,\lambda)$ is the surface spherical harmonic of n-th degree, associated with ξ. We can thus write

$$(u_a)_n = \alpha_n \xi_n , \tag{3.19a}$$

and

$$u_a = \sum_n \alpha_n \xi_n \tag{3.19b}$$

where

$$\alpha_n = [3/(2n+1)] \times (\text{average water density/average earth density}). \tag{3.19c}$$

The expression for α_n agrees with [PH], page 393, if one disregards an apparent typographical error in that the first factor therein is written as $(3/2n+1)$.

Effect of the ocean tidal loading on the ocean bottom. The deformation of the ocean bottom due to the ocean tidal loading is symbolized by u_ℓ. In our context, it is a function on a sphere which can be expanded in surface spherical-harmonics as

$$u_\ell = \sum_n (u_\ell)_n . \tag{3.20}$$

As is explained in [V], pages 21 and 22, the displacement u_a serves as a norm for u_ℓ, giving rise to the following load numbers (sometimes called Love load numbers) for all possible wave numbers n:

$$h'_n = (u_\ell)_n/(u_a)_n .$$

This together with (3.18b) and (3.20) yield

$$u_\ell = \sum_n h'_n(u_a)_n = (1/g) \sum_n h'_n(V_s)_n = (R^3\rho/M_e) \sum_n h'_n \iint_\Omega \xi P_n (\cos\psi)\, d\Omega' , \tag{3.21}$$

where the second equality corresponds to (64) in [V]. If, in the last expression, the summation and the integration are interchanged one has

$$u_\ell = (R^3\rho/M_e)\iint_\Omega \xi \sum_n h'_n P_n(\cos\psi)\, d\Omega' = R^2\rho \iint_\Omega \xi\, U'(\psi)\, d\Omega' , \tag{3.22a}$$

$$U'(\psi) = (R/M_e) \sum_n h'_n P_n(\cos\psi) . \tag{3.22b}$$

The first equality in (3.22a) corresponds to (65) and (66) of [V], the second equality serves as another building block in (A10) of [E], and (3.22b) is the definition appearing in (A5) of [E].

If $(u_a)_n$ in the first equivalence of (3.21) is taken from (3.19a), we write at once:

$$u_\ell = \sum_n h'_n \alpha_n \xi_n , \tag{3.23}$$

which corresponds to the "ocean induced vertical component of the solid earth tide" as it appears on page 393 of [PH].

<u>Effect of the ocean tidal loading on the potential</u>. The disturbing potential due to the deformation u_ℓ just treated is denoted as V_b. Similar to (3.17), it gives rise to the displacement u_i of an equipotential surface,

$$u_i = V_b/g . \tag{3.24}$$

This displacement can again be expanded in surface spherical-harmonics:

$$u_i = \sum_n (u_i)_n . \tag{3.25}$$

The statement that followed (3.20) now applies to u_i in conjunction with another kind of load numbers:

$$k'_n = (u_i)_n/(u_a)_n .$$

This together with (3.18b) and (3.25) yield

$$u_i = \sum_n k'_n (u_a)_n = (1/g) \sum_n k'_n (V_s)_n = (R^3\rho/M_e) \sum_n k'_n \iint_\Omega \xi \, P_n (\cos\psi) \, d\Omega' \ , \tag{3.26}$$

where the second equality corresponds to (64) of [V]. Upon interchanging the summation and the integration in the last expression we obtain

$$u_i = (R^3\rho/M_e) \iint_\Omega \xi \sum_n k'_n P_n (\cos\psi) \, d\Omega' = (R^2\rho/g) \iint_\Omega \xi \, \Phi'(\psi) \, d\Omega' \ , \tag{3.27a}$$

$$\Phi'(\psi) = (Rg/M_e) \sum_n k'_n P_n (\cos\psi) \ . \tag{3.27b}$$

The last expression in (3.27a) serves as a third building block in (A10) of [E], and (3.27b) is the same definition as in (A5) of [E]. From (3.24) and (3.27a) we also have

$$V_b = gu_i = R^2\rho \iint_\Omega \xi \, \Phi'(\psi) \, d\Omega' \ . \tag{3.28}$$

If $(u_a)_n$ in the first equivalence of (3.26) is taken from (3.19a), it follows that

$$u_i = \sum_n k'_n \alpha_n \xi_n \ . \tag{3.29}$$

Upon utilizing this form for V_b , one obtains

$$V_b = g \sum_n k'_n \alpha_n \xi_n \ , \tag{3.30}$$

which corresponds to the second part of the "ocean induced potential at the mean sea surface" as presented on page 393 of [PH].

The total tidal potential causing the water to depart from its average level is composed of the astronomical tidal potential Γ, of the additional tidal potential $k\Gamma$ due to the earth deformation, of V_s due to the self-gravitation, and of V_b due to the ocean loading:

$$\Gamma_{tot} = (1+k)\Gamma + V_s + V_b \ . \tag{3.31}$$

When written in terms of (3.17) and (3.24), this becomes

$$\Gamma_{tot} = (1+k)\Gamma + gu_a + gu_i \ , \tag{3.31'}$$

which, with (3.19b) and (3.29), also is

$$\Gamma_{tot} = (1+k)\Gamma + g \sum_n (1+k_n') \alpha_n \xi_n \ ;$$

this formulation corresponds to that on page 393 of [PH]. If the water level adjusted itself instantaneously to an equipotential surface implied by (3.31'), its height above the mean surface would be

$$\xi_s' = \Gamma_{tot}/g = (1+k)\Gamma/g + u_a + u_i \ . \tag{3.32}$$

This quantity would not be the same as ξ_s obtained from the solution of the LTE, involving the forces of friction, viscosity, etc. This fact is reflected by the prime attributed to ξ_s in (3.32).

The departure of the ocean bottom from its average value is given as

$$\xi_b = h\Gamma/g + u_\ell \ . \tag{3.33}$$

If u_ℓ is written as in (3.23), equation (3.33) becomes

$$\xi_b = h\Gamma/g + \sum_n h'_n \alpha_n \xi_n ,$$

which again corresponds to the formulation on page 393 of [PH]. Under the simplification (3.32), the ocean tide would be

$$\xi' = \xi'_s - \xi_b = (1 + k - h)\Gamma/g + (u_a + u_i - u_\ell). \tag{3.34}$$

If, in addition, we also ignored the self-gravitation and the ocean loading effects, i.e., u_a, u_i and u_ℓ in (3.32)-(3.34),we would have

$$\xi''_s = (1 + k)\Gamma/g ,$$

$$\xi'_b = h\Gamma/g ,$$

$$\xi'' = \xi''_s - \xi'_b = (1 + k - h)\Gamma/g .$$

But this is the simplified tidal model used in [B] and recapitulated in the previous section, where the earth deformation corresponds to the above ξ'_b , the "geocentric tide" corresponds to ξ''_s , and the "measured tide" corresponds to ξ'' . The above process of successively greater simplifications has thus brought into focus the assumptions and simplifications present in the model of [B].

Most of the quantities just discussed are shown in Figure 1. The figure represents the realistic tidal model and involves the solution of the LTE; this fact is illustrated by the presence of the quantity y. A brief description of the quantities in Figure 1 follows:

Γ/g = equilibrium tide ,

y = quantity resulting from the solution of the LTE ,

$k\Gamma/g$ = additional tide (due to the earth deformation $h\Gamma/g$) ,

u_i = V_b/g = displacement due to the ocean tidal loading on the ocean bottom (i.e., due to u_ℓ) ,

u_a = V_s/g = displacement due to the self-gravitation of tidal waters ;

$h\Gamma/g$ = earth deformation (called also solid earth tide) ,

u_ℓ = ocean bottom deformation due to the ocean tidal loading ;

ξ_s = surface tide ,

ξ_b = bottom tide ,

ξ = ocean tide .

From Figure 1 it follows that

$$\xi_s = (1+k)\Gamma/g + y + u_a + u_i \ , \tag{3.35}$$

$$\xi_b = h\Gamma/g + u_\ell , \tag{3.36}$$

$$\xi = \xi_s - \xi_b = (1+k-h)\Gamma/g + y + (u_a + u_i - u_\ell) \ . \tag{3.37}$$

It is also apparent from the figure that

$$\xi = h(\text{instant.}) - h(\text{ave.}) \ . \tag{3.37'}$$

Without the presence of y, (3.35)-(3.37) would reduce to (3.32 - 3.34).

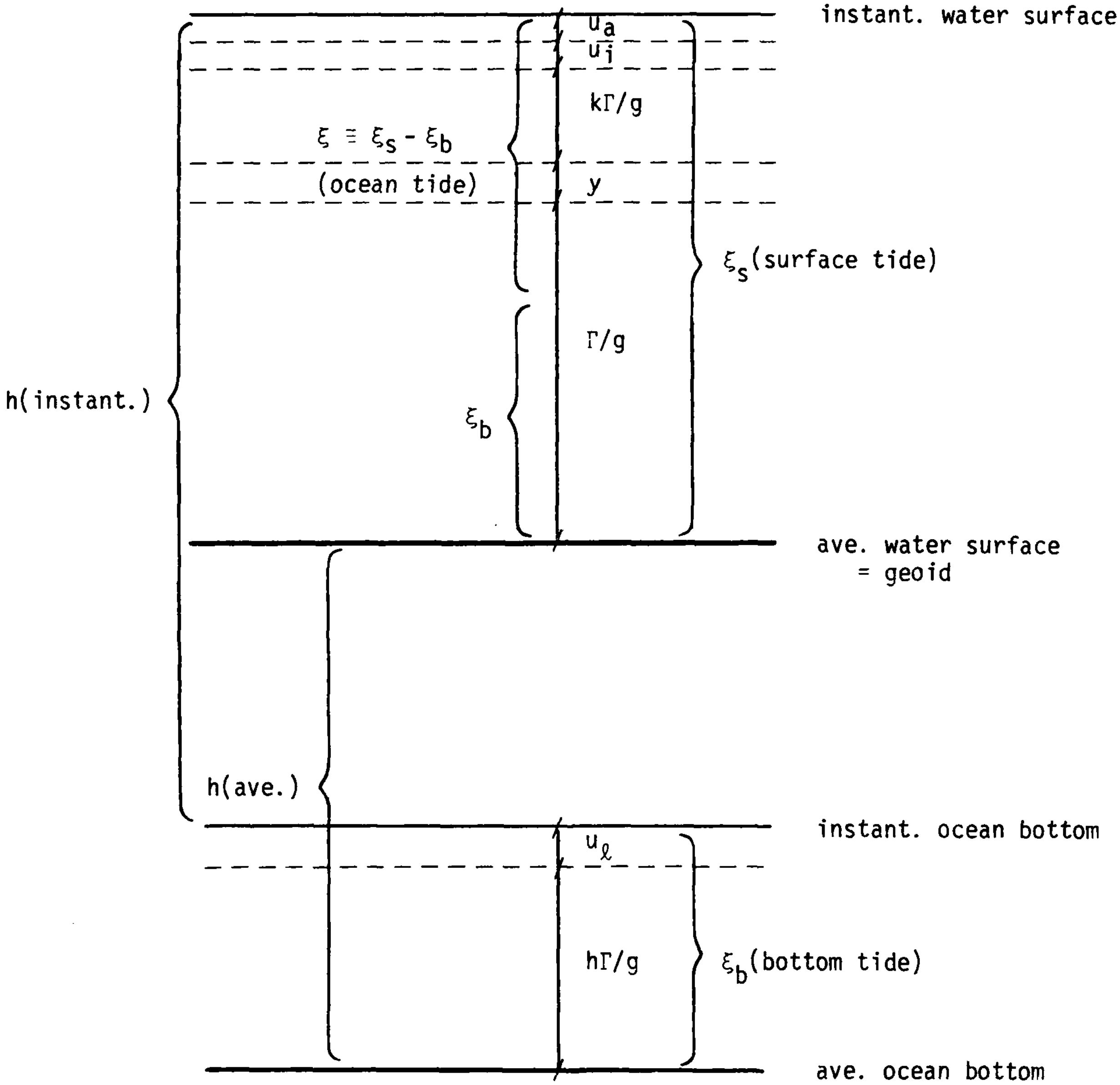

Figure 1
Schematic display of various quantities related to the tidal phenomenon

At this point, three remarks can be made upon examining Figure 1. First, identifying the average water surface with the geoid indicates a few approximations (such as considering the average water surface to be equipotential, which is not exactly fulfilled in several oceanic areas) and, more importantly, it indicates that the permanent tide is ignored in the present considerations. However, this tidal effect, which essentially transforms the "static geoid" (of interest) into the above geoid, is treated separately. In fact, all the pertinent long-period tidal effects are envisioned to be treated separately along the lines of the simplified model; the realistic tidal model as adapted from [E] was developed only for the principal diurnal and semidiurnal tides. We note that in [S], pages 165 and 166, and in [Estes, 1977], Figure 1, the same identification (ave. water surface = geoid) was made as in our Figure 1.

Second, we remark that for $\xi>0$, one must have $u_\ell<0$ and thus also $u_i<0$; the response of u_i to u_ℓ is qualitatively somewhat similar to the response of the additional tide ($k\Gamma/g$) to the earth deformation ($h\Gamma/g$). On the other hand, if $\xi>0$, the disturbing potential increases (see e.g. the illustrative case 3.9a) and thus the corresponding geoid undulation also increases, i.e., one has $u_a>0$. Therefore, u_i and u_ℓ have the opposite signs from that of u_a and should have been directed downwards in Figure 1 where ξ and u_a have an upward sense. In Figure 1 of [S], u_ℓ is indeed represented in this manner and is appropriately called the solid earth dip, ξ^{eo} (i.e., the earth-dip response to the oceanic tidal load ξ). But this is merely a matter of convention since in this reference ξ_b is expressed essentially as $h\Gamma/g$ - (positive quantity, ξ^{eo}) while in the present study ξ_b is given as $h\Gamma/g$ + (negative quantity, u_ℓ). A similar comment would also apply to u_i, as may be gathered upon consulting equation (9) of the same reference. Figure 10 of [V] where k_n' and h_n' are seen to be negative for all values of n helps us to verify, in a plausible way, that the quantities u_i and u_ℓ have the opposite signs of u_a .

The last remark concerns the notations used in the figure as well as throughout the text. As stated in [V], page 10, only the surface values h_2 , k_2 of Love numbers h_n , k_n are used in geodetic applications, in which case the symbols h , k are often employed. These symbols have been adopted in the present study. It was demonstrated on pages 3 and 4 of [V] that the astronomical tidal potential, Γ, can for most practical purposes be equated with Γ_2 of the development in terms of Legendre polynomials. It should thus be borne in mind that in our context Γ, $k\Gamma$ and $h\Gamma$ could replace Γ_2 , $k_2\Gamma_2$ and $h_2\Gamma_2$ or $\sum_n \Gamma_n$, $\sum_n k_n \Gamma_n$ and $\sum_n h_n \Gamma_n$, respectively, used in some references.

In the upcoming derivation, the horizontal components of the force corresponding to (3.31) or (3.31') will be needed along the λ- and ϕ-coordinate lines. Since one can write for this force:

$$\overline{F}' = \text{grad}\ \Gamma_{tot} ,$$

the required components are

$$F_\lambda' = [1/(R\cos\phi)]\partial\Gamma_{tot}/\partial\lambda = [1/(R\cos\phi)][(1+k)\partial\Gamma/\partial\lambda + g\partial(u_a+u_i)/\partial\lambda] , \tag{3.38a}$$

$$F_\phi' = (1/R)\ \partial\Gamma_{tot}/\partial\phi = (1/R)\ [(1+k)\ \partial\Gamma/\partial\phi + g\partial(u_a+u_i)/\partial\phi]. \tag{3.38b}$$

These components will be used, together with the components F_λ'' and F_ϕ'' of $\overline{F}''$, the force of friction and viscosity, in the LTE in the form (A1) of [E]. The components of $\overline{F}''$ will be transcribed from this reference, but otherwise the final form of these equations will be rederived independently, for the purpose of scrutinizing the quantity ξ_s needed for a refinement of altimetric applications. The model of [E] is advantageous especially because the tidal elevations ξ for all the diurnal and semidiurnal constituents of interest are developed in terms of spherical-harmonic tidal coefficients, tailored for use in the first phase of satellite altimetry adjustment (global adjustment) with the spherical-harmonic potential coefficients as parameters.

The first two (out of three) LTE presented in (A1) of [E] read

$$\partial u/\partial t - 2\omega \sin\phi \, v = -[g/(R \cos\phi)]\partial\xi_s/\partial\lambda + F_\lambda' + F_\lambda'' , \tag{3.39a}$$

$$\partial v/\partial t + 2\omega \sin\phi \, u = - (g/R)\partial\xi_s/\partial\phi + F_\phi' + F_\phi'' , \tag{3.39b}$$

where, in addition to the symbols already defined, u and v are the eastward and northward components of the water velocity and ω is the rotational velocity of the earth. These equations involve not only the tidal generating potential (it gives rise to F_λ', F_ϕ'), but also the forces of friction and viscosity (see F_λ'' , F_ϕ'' mentioned earlier) and the Coriolis force (it is associated with ω on the left-hand sides of 3.39 a,b). The third equation in (A1) involves the water depth and it will be transcribed directly into the final formulas. The three equations (A1) correspond to equations (5) - (7) of [Lisitzin, 1974].

Next, F_λ' and F_ϕ' from (3.38 a,b) are substituted into (3.39 a,b), together with ξ_s, expressed from Figure 1 as

$$\xi_s = \xi + \xi_b = \xi + h\Gamma/g + u_\ell . \tag{3.40}$$

This results in

$$\partial u/\partial t - 2\omega \sin\phi\ v = [1/(R \cos\phi)]\ [-g\partial\xi/\partial\lambda + (1 + k - h)\ \partial\Gamma/\partial\lambda + g\partial(u_a + u_i - u_\ell)/\partial\lambda] + F''_\lambda\ , \tag{3.41a}$$

$$\partial v/\partial t + 2\omega \sin\phi\ u = (1/R)[-g\partial\xi/\partial\phi + (1 + k - h)\partial\Gamma/\partial\phi + g\partial(u_a + u_i - u_\ell)/\partial\phi] + F''_\phi\ . \tag{3.41b}$$

According to (3.19b), (3.23) and (3.29), we could write

$$u_a + u_i - u_\ell = \sum_n (1 + k'_n - h'_n)\alpha_n \xi_n\ .$$

However, to relate this quantity to the results in [E], we use (3.17'), (3.22a) and (3.27a):

$$u_a + u_i - u_\ell = R^2\rho \iint_\Omega \xi\ [(R/M_e)(2 \sin \tfrac{1}{2}\psi)^{-1} + (1/g)\Phi'(\psi) - U'(\psi)]d\Omega'. \tag{3.42}$$

In agreement with [E], Γ' is defined at (ϕ,λ,t) as

$$\Gamma' \ = \ g(u_a + u_i - u_\ell)\ . \tag{3.43a}$$

With the aid of (3.42), equation (3.43a) reads

$$\Gamma' \ = \ \iint_\Omega \xi * \text{"Green"}\ d\Omega'\ , \tag{3.43b}$$

where the Green's function is

$$\text{"Green"} = R^2\rho[Rg/M_e)(2 \sin \tfrac{1}{2}\psi)^{-1} + \Phi'(\psi) - gU'(\psi)]\ . \tag{3.43c}$$

The last two equations correspond to the result on pages 106 and 107 of [E], especially equation (A10).

The last step consists of utilizing (3.43a) in (3.41 a,b) and transcribing F_λ'', F_ϕ'' from [E], page 107; the two equations are then complemented by the third equation in (A1), page 93 of [E]. The result is

$$\partial u/\partial t - 2\omega \sin\phi\, v = [1/(R\cos\phi)]\ [-g\partial\xi/\partial\lambda + (1+k-h)\ \partial\Gamma/\partial\lambda + \partial\Gamma'/\partial\lambda] - C_r(u^2+v^2)^{\frac{1}{2}}\ u/H + C_{hv}\Delta u\ , \tag{3.44a}$$

$$\partial v/\partial t + 2\omega \sin\phi\, u = (1/R)[-g\partial\xi/\partial\phi + (1+k-h)\ \partial\Gamma/\partial\phi + \partial\Gamma'/\partial\phi] - C_r(u^2+v^2)^{\frac{1}{2}}\ v/H + C_{hv}\Delta v\ , \tag{3.44b}$$

$$\partial\xi/\partial t = -[1/(R\cos\phi)]\ [\partial(Hu)/\partial\lambda + \partial(H v\cos\phi)/\partial\phi]\ , \tag{3.44c}$$

where Δ is the horizontal Laplacian, C_r and C_{hv} are the coefficients of friction and horizontal eddy viscosity, and

$$H = h(\text{ave.}) + \xi \equiv h(\text{instant.})\ .$$

Detailed information about the numerical solution of equations (3.44 a-c) can be found on pages 106-117 of [E].

After ξ has been solved for, the quantity ξ_s, directly related to satellite altimetry, can be expressed as in (3.40), namely

$$\xi_s = \xi + h\Gamma/g + u_\ell\ , \tag{3.45a}$$

where

$$h\Gamma/g = h * (\text{equilibrium tide})\ . \tag{3.45b}$$

In practical adjustments of altimeter data, the quantity u_ℓ is envisioned either to be neglected or to be expressed very approximately as some multiple of ξ. The latter choice could be carried out in conjunction with the M_2 constituent for which ξ in [E] is expressed through the solution of (3.44 a-c) in an iterative process (in this case Γ is made to correspond to M_2). The remaining semidiurnal and diurnal constituents are treated, in this reference, in a more approximate fashion by ignoring Γ'. In order to express ξ_s as accurately as practicable the quantities u_a , u_i and u_ℓ as well as their combinations $(u_a + u_i)$ and, especially, $u_a + u_i - u_\ell = \Gamma'/g$ from (3.43a) will be examined in the light of the numerical solutions presented in [PH] and [E].

The contour maps in Figures 1, 3 and 5 of [PH] depict ξ for the constituents M_2, S_2 and K_1, respectively. The contour maps of $(u_a + u_i)$ for these constituents are shown in Figures 10, 11 and 12, and the contour maps of u_ℓ are similarly shown in Figures 13, 14 and 15 of this reference. When examining the results for u_ℓ we imagine the phase changed by 180° and, accordingly, the sign changed. Upon inspection, the quantity $-u_\ell$ is then seen to be essentially "in phase" with ξ, consistent with our earlier discussion.

For the M_2 constituent, ξ ranges mostly within 25 - 50 cm (exceptionally, it reaches over 1m), while $(u_a + u_i)$ ranges mostly within 1-3 cm (exceptionally 4 cm), and $-u_\ell$ ranges mostly within 1-4 cm (in two coastal areas it reaches about 6 cm). The phases as well as the highs and lows of these quantities have similar characteristics over most of the global oceans. From this inspection one can draw very approximate tentative conclusions:

$$u_a + u_i \lesssim 0.05\xi, \qquad -u_\ell \gtrsim 0.05\xi; \qquad u_a + u_i - u_\ell \simeq 0.10\xi . \qquad (3.46)$$

Nearly the same conclusions can be reached also for the S_2 and K_1 constituents. In the case of S_2 , ξ is mostly in the 10 - 30 cm range (exceptionally 40 cm),

$(u_a + u_i)$ is in the 0.5 - 1.5 cm range and $-u_\ell$ is mostly in the 0.5 - 1.5 cm range (exceptionally 2 cm). The ranges for the K_1 constituent are similar to these.

Next, [E] is consulted with regard to the M_2 constituent for which contour maps are presented. Figures A1 (for ξ), A8 (for $u_a + u_i - u_\ell$), A9 (for u_a) and A10 (for $-u_\ell$) allow the following observations: ξ is mostly in the 10 - 50 cm range (exceptionally 80 cm in the Equatorial Pacific and east of Australia); $u_a + u_i - u_\ell$ is mostly in the 1 - 5 cm range (exceptionally 12 cm in the Equatorial Pacific); and u_a and u_ℓ, both having very similar characteristics, are typically in the 1 - 3 cm range (exceptionally 7 cm in the Equatorial Pacific). There is a generally good agreement in the phases and in the relative magnitudes of these four quantities. A visual inspection leads to the following approximate conclusions:

$$u_a \gtrsim 0.05\xi \ , \qquad -u_\ell \gtrsim 0.05\xi \ ; \qquad u_a + u_i - u_\ell \simeq 0.10\xi \ . \tag{3.47}$$

The last relations in (3.46) and (3.47) are supported by the result described on pages 170 and 171 of [S]. Transcribed into our notations it states that

$$u_a + u_i - u_\ell \equiv \Gamma'/g \simeq 0.10\xi \ ,$$

obtained "after evaluating the Green's function representation of the three oceanic tidal load effects" (page 171). On the same page of [S] it is also stated: "The author's computations supported the marginal effects of oceanic tidal loading found by Estes (1977)."

From the above, especially from the contour maps associated with the M_2 constituent, the various effects can be estimated very approximately as follows:

$$u_a \simeq 0.06\xi , \qquad u_i \simeq -0.02\xi , \qquad u_a + u_i \simeq 0.04\xi ;$$

$$-u_\ell \simeq 0.06\xi ; \tag{3.48}$$

$$u_a + u_i - u_\ell \simeq 0.10\xi .$$

Of these, (3.48) is the most important, allowing us to express ξ_s from (3.45 a,b) as

$$\xi_s = 0.94\xi + h * \text{(equilibrium tide)} . \tag{3.49}$$

This formula could be used in conjunction with any constituent whose ξ is computed rigorously as in (3.44 a-c). In [E] such a computation was performed for the M_2 constituent. As was already mentioned, the self-gravitation and the ocean loading effects were neglected in [E] (Γ' was disregarded) during the computations of ξ for the remaining diurnal and semidiurnal constituents. To be consistent with this procedure when adopting the spherical-harmonic tidal coefficients from [E], we neglect u_ℓ in (3.45a) applied to these constituents, in which case (3.45 a,b) yield

$$\xi_s = \xi + h * \text{(equilibrium tide)} . \tag{3.50}$$

The above analysis suggests the use of (3.49) in conjunction with M_2 and the use of (3.50) in conjunction with all othe other diurnal and semidiurnal constituents considered. In fact, (3.50) could have been designed for M_2 as well since, according to the earlier results taken from [PH] and [E], the magnitude of u_ℓ is usually well below the 5 cm level and the effect of u_ℓ could also qualify as being only marginal. However, we prefer to use a more accurate model (3.49) because not only the tidal amplitudes, but also the phase angles are now considered to be subject to adjustment, and the model is nonlinear in the phase angles. Hence the best possible initial values should be sought for the parameters appearing in the linearized observation equations. Although one could argue that the value ξ is not significantly different from the value 0.94ξ for this purpose, it is equally true that the more accurate model for M_2 represented by (3.49) is as easy to handle as the model represented by (3.50).

As was done by Estes [1980] for the desired diurnal and semidiurnal constituents, the ocean tide ξ can be expressed by means of spherical-harmonic tidal coefficients encompassing, through a least-squares fit, the solution of the LTE. For a given tidal constituent, these coefficients lead to an expansion similar to equation (3) of [Estes, 1980]. This formulation is described with the aid of the following model:

$$\xi_j = A_j \cos(\alpha_j - \psi_j) \equiv A_j \cos\psi_j \cos\alpha_j + A_j \sin\psi_j \sin\alpha_j \;, \qquad (3.51)$$

where

ξ_j = constituent height observable by tidal gauges,

α_j = Greenwich argument of the constituent,

$A_j \equiv A_j(\phi,\lambda)$ = amplitude of the constituent,

$\psi_j \equiv \psi_j(\phi,\lambda)$ = phase angle of the constituent.

The longitude (λ) of the place of observation is not explicitly needed since it is included in ψ_j. The angles α_j are thus computed as in (2.43a) - (2.44d).

Equation (3.51) is reformulated to read

$$\xi_j = a_j \cos\alpha_j + b_j \sin\alpha_j \ , \tag{3.52a}$$

where

$$a_j \equiv a_j(\phi,\lambda) = A_j \cos\psi_j = \sum_n \sum_m (a_{j_{nm}} \cos m\lambda + b_{j_{nm}} \sin m\lambda) P_{nm}(\sin\phi), \tag{3.52b}$$

$$b_j \equiv b_j(\phi,\lambda) = A_j \sin\psi_j = \sum_n \sum_m (c_{j_{nm}} \cos m\lambda + d_{j_{nm}} \sin m\lambda) P_{nm}(\sin\phi), \tag{3.52c}$$

from which it follows that

$$A_j = (a_j^2 + b_j^2)^{1/2} \ , \tag{3.52d}$$

$$\cos\psi_j = a_j/A_j \ , \qquad \sin\psi_j = b_j/A_j \ . \tag{3.52e}$$

In these formulas $a_{j_{nm}}$, etc., are the spherical-harmonic tidal coefficients of degree and order (n,m) belonging to the constituent j, and $P_{nm}(\sin\phi)$ are the associated Legendre functions in the argument $\sin\phi$, ϕ being the geocentric latitude. In the next chapter, the above formulation will be used in conjunction with the model for ξ_s seen previously.

4. TIDAL ADJUSTMENT

4.1 Inclusion of Tidal Effects in the Altimetry Adjustment Model

The basic model equation of satellite altimetry was written in equation (3.1) of [Blaha, 1979] as

$$H = R - r + d ,$$

where H represents the altimetry, R is the distance from the geocenter to the satellite at the time of observation, d is a correction, always smaller than 5 m for the satellite altitude under 1,000 km as described e.g. on page 28 of [Blaha, 1977] or in [Blaha, 1977'], and r is the distance from the geocenter to a sub-satellite point on the sea surface; it is given on page 15 of [Blaha, 1979] as

$$r = r' + N ,$$

where r' is the corresponding distance to the (geocentric) reference ellipsoid and N represents the geoid undulation. The main feature of an earlier approach consisted in expressing N (and thus r) in terms of the geoidal parameters only, as if the measured sea surface coincided with the geoid. Although this model deficiency was of little consequence in past adjustments of GEOS-3 altimetry, it will be at least partly removed from the SEASAT altimetry model by separating N into two parts:

$$N = N' + N'' ,$$

where N' is expressed in terms of the geoidal parameters as before, but where N" should ideally represent the separation between the geoid and the measured sea surface.

In this analysis, N" is expressed in terms of the 11 tidal constituents considered. A physically meaningful model can be conceived by allowing some or all of the tidal amplitudes and phase angles to adjust within the overall adjustment of satellite altimetry. Due to a high degree of uncertainty in the long-period constituents (see, e.g., the "empirical factor" mentioned in [B]), the model for these constituents should be linear. This will be ensured by subjecting only the tidal amplitudes to adjustment, not the phase angles. On the other hand, since the diurnal and semidiurnal tidal effects are assumed to be known to a better degree of accuracy, both these kinds of parameters will enter the adjustment.

Accordingly, 18 tidal parameters will be added to the adjustment of SEASAT altimetry: 4 amplitude parameters (P_j) due to the long-period constituents, $j=A_0$, *Mf*, *Mm*, *SSa*; 3 amplitude parameters (P_j) and 3 phase-angle parameters (Q_j) due to the diurnal constituents, $j=K_1$, O_1, P_1; and 4 amplitude parameters (P_j) and 4 phase-angle parameters (Q_j) due to the semidiurnal constituents, $j=M_2$, S_2, N_2, K_2. In considering that the global adjustment contains some 225 terrestrial parameters (spherical-harmonic potential coefficients in a 14,14 truncated model) and the regional adjustment may contain a comparable number of parameters (point-mass magnitudes), the addition of 18 parameters at either level does not increase the computer burden by a great amount, both from the run-time and storage requirement standpoint.

4.2 Observation Equations

In agreement with Section 4.1, N", representing that part of an "instantaneous geoid undulation" sensed by the altimeter which can be attributed to the tidal constituents under consideration, can be expressed as

$$N'' = (h_{A_0} + h_{Mf} + h_{Mm} + h_{SSa}) + (h_{K_1} + h_{O_1} + h_{P_1}) + (h_{M_2} + h_{S_2} + h_{N_2} + h_{K_2}) \ , \tag{4.1}$$

where the constituent heights h_j are the a priori values (not equilibrium in general). The first group identified by the parentheses represents the long-period constituents, the second group represents the diurnal constituents and the third group represents the semidiurnal constituents.

From the above, the tidal part of an altimeter observation equation is formed in a familiar fashion through a *linearization process*. As indicated earlier, there will be only one adjustable parameter per long-period constituent, namely the amplitude parameter (P_j); the diurnal and semidiurnal constituents will comprise two adjustable parameters each (P_j, Q_j). The parameter P_j represents a relative amplitude correction (i.e., an amplitude correction divided by the amplitude itself) while the parameter Q_j represents a correction in degrees to the phase angle. The observation equations are developed in terms of the available a priori values for h_j, i.e., the linearization proceeds with the initial values $P_j=0$, $Q_j=0$. When evaluated using the a priori information, (4.1) represents the tidal contribution to the constant term in an altimeter observation equation. Upon denoting the pertinent coefficients by p_j and q_j in a self-evident manner, the total tidal contribution to an altimeter observation equation can be symbolized by

$$N''^a = (p_{A_o} P_{A_o} + p_{Mf} P_{Mf} + p_{Mm} P_{Mm} + p_{SSa} P_{SSa}) + (p_{K_1} P_{K_1} + q_{K_1} Q_{K_1} + p_{O_1} P_{O_1} + q_{O_1} Q_{O_1}$$
$$+ p_{P_1} P_{P_1} + q_{P_1} Q_{P_1}) + (p_{M_2} P_{M_2} + q_{M_2} Q_{M_2} + p_{S_2} P_{S_2} + q_{S_2} Q_{S_2} + p_{N_2} P_{N_2} + q_{N_2} Q_{N_2}$$
$$+ p_{K_2} P_{K_2} + q_{K_2} Q_{K_2}) + N'' , \tag{4.2}$$

where the superscript "a" indicates an adjusted value and where N" is evaluated by (4.1) as stated above.

As is apparent from Section 4.1, an original altimeter observation equation (without the tidal part) should now be modified by adding -N" to the original constant term, augmenting the row of coefficients by

$$[-p_{A_o}, -p_{Mf}, -p_{Mm}, -p_{SSa}; \ -p_{K_1}, -q_{K_1}, -p_{O_1}, -q_{O_1}, -p_{P_1}, -q_{P_1};$$
$$-p_{M_2}, -q_{M_2}, -p_{S_2}, -q_{S_2}, -p_{N_2}, -q_{N_2}, -p_{K_2}, -q_{K_2}], \tag{4.3}$$

and augmenting the column of parameters by

$$[P_{A_o}, P_{Mf}, P_{Mm}, P_{SSa}; \ P_{K_1}, Q_{K_1}, P_{O_1}, Q_{O_1}, P_{P_1}, Q_{P_1};$$
$$P_{M_2}, Q_{M_2}, P_{S_2}, Q_{S_2}, P_{N_2}, Q_{N_2}, P_{K_2}, Q_{K_2}]^T . \tag{4.4}$$

The formula (4.2) can also be written as

$$N''^a = \sum_j h_j^a ,$$

where the adjusted constituent heights are

$$h_j^a = p_j P_j + q_j Q_j + h_j . \tag{4.5}$$

The coefficients p_j for the long-period constituents will be derived first (the coefficients q_j are immaterial in this case), followed by a derivation of the coefficients p_j, q_j for the diurnal and semidiurnal constituents.

Long-period constituents. The model equation for these constituents corresponds here to the simplified model of Section 3.1; in agreement with the latter and with [B], Section 3.3, we can write

$$h_j = A_j \cos\alpha_j \ , \tag{4.6a}$$

$$A_j = c\,A'_j \ , \tag{4.6b}$$

$$c = (1+k)e \ , \tag{4.6c}$$

where

A'_j = equilibrium amplitude ,

α_j = Greenwich or local argument (longitude is irrelevant here) ,

$1+k \simeq 1.29$ (k is a Love number) ,

e = adopted "empirical factor" .

From (4.6a) one has at once:

$$dh_j = dA_j \cos\alpha_j = (A_j \cos\alpha_j)(dA_j/A_j) \equiv h_j P_j \ .$$

Accordingly,

$$h_j^a = h_j + dh_j = h_j P_j + h_j \ , \tag{4.7a}$$

and thus

$$p_j = h_j \ . \tag{4.7b}$$

The next step consists in evaluating h_j. Under idealized conditions (equilibrium tide allowing for the earth's deformation, see page 49 of [B]), in conjunction with (4.6c) one would have

$$e' \simeq 1, \qquad c \simeq 1+k \simeq 1.29 .$$

The value used by Lisitzin [1974] -- but attributed to another source -- corresponds to

$$e'' \simeq 3.7 , \quad c \simeq (1+k) \times 3.7 \simeq 4.8 ,$$

which was also adopted in [B]. However, the above value for e" is highly uncertain and could be too high. To be on the conservative side, we may adopt an average value,

$$e = \frac{1}{2}(e' + e'') \simeq 2.35 ,$$

and thus

$$c \simeq (1+k) \times 2.35 \simeq 3.03 . \tag{4.8}$$

For the equilibrium amplitudes we have from Section 2.1:

$$A'_{A_o} = 0.1073\,\mathrm{m}\,(1 - 3\sin^2\phi) ,$$

$$A'_{Mf} = 0.0131\,\mathrm{m}\,(1 - 3\sin^2\phi) ,$$

$$A'_{Mm} = 0.0125\,\mathrm{m}\,(1 - 3\sin^2\phi) ,$$

$$A'_{SSa} = 0.0097\,\mathrm{m}\,(1 - 3\sin^2\phi) .$$

If these results together with c in (4.8) are utilized in (4.6 a,b), one obtains

$$h_{A_0} = 0.325\,\mathrm{m}\,(1 - 3\sin^2\phi) \ , \tag{4.9a}$$

$$h_{Mf} = 0.040\,\mathrm{m}\,(1 - 3\sin^2\phi)\cos\alpha_{Mf} \ , \tag{4.9b}$$

$$h_{Mm} = 0.038\,\mathrm{m}\,(1 - 3\sin^2\phi)\cos\alpha_{Mm} \ , \tag{4.9c}$$

$$h_{SSa} = 0.029\,\mathrm{m}\,(1 - 3\sin^2\phi)\cos\alpha_{SSa} . \tag{4.9d}$$

The arguments α_j corresponding to SEASAT observational epoch appear in (2.42 a-d) while α_j' appear in (2.39 a-c).

Diurnal and semidiurnal constituents. According to (3.49) and (3.50), the model equation is

$$h_j = c'\xi_j + h * (\text{equilibrium tide})_j \ , \tag{4.10}$$

where

$c' = 0.94$ ... M_2 constituent ,

$c' = 1.00$ ... the other diurnal and semidiurnal constituents;

$$\xi_j = a_j \cos\alpha_j + b_j \sin\alpha_j \ \text{... ocean tide} \ , \tag{4.10'}$$

$a_j = A_j \cos\psi_j$, $b_j = A_j \sin\psi_j$ ... values computed from spherical-harmonic tidal coefficients as in (3.52 b,c) ,

α_j = Greenwich argument, as in (2.43a) - (2.44d) with α_j' in (2.40a) - (2.41d);

$h \simeq 0.62$ (a Love number) ;

$$(\text{equilibrium tide})_j = A'_j \cos\tilde{\alpha}_j , \qquad (4.10'')$$

A'_j = equilibrium amplitude, as presented below,

$\tilde{\alpha}_j$ = local argument, as in (2.45 a-g).

The A'_j for the diurnal constituents are gathered from Section 2.2 as

$$A'_{K_1} = 0.125\,\text{m} \sin 2\phi ,$$

$$A'_{O_1} = 0.081\,\text{m} \sin 2\phi ,$$

$$A'_{P_1} = 0.047\,\text{m} \sin 2\phi .$$

For the semidiurnal constituents we have from Section 2.3:

$$A'_{M_2} = 0.252\,\text{m} \cos^2\phi ,$$

$$A'_{S_2} = 0.113\,\text{m} \cos^2\phi ,$$

$$A'_{N_2} = 0.049\,\text{m} \cos^2\phi ,$$

$$A'_{K_2} = 0.023\,\text{m} \cos^2\phi .$$

Since $(\text{equilibrium tide})_j$ is not adjustable, one has

$$dh_j = c'\, d\xi_j . \qquad (4.11)$$

From (4.10') and the relation that followed we obtain

$$\xi_j = A_j \cos\psi_j \cos\alpha_j + A_j \sin\psi_j \sin\alpha_j ,$$

$$\begin{aligned} d\xi_j = {} & (A_j \cos\psi_j \cos\alpha_j + A_j \sin\psi_j \sin\alpha_j)(dA_j/A_j) \\ & + (1/\rho)(-A_j \sin\psi_j \cos\alpha_j + A_j \cos\psi_j \sin\alpha_j)\rho\, d\psi_j ; \end{aligned}$$

this is

$$d\xi_j = (a_j \cos\alpha_j + b_j \sin\alpha_j)P_j + (1/\rho)(-b_j \cos\alpha_j + a_j \sin\alpha_j)Q_j \ , \qquad (4.12)$$

where

$$P_j \equiv dA_j/A_j, \qquad Q_j \equiv \rho \, d\psi_j \equiv d\psi_j^0 \ ,$$

with $\rho = 57.295780$ transforming the correction to the phase angle from radians to degrees. Accordingly, (4.10), (4.11), (4.12) yield

$$h_j^a = c'(a_j \cos\alpha_j + b_j \sin\alpha_j)P_j + (c'/\rho)(a_j \sin\alpha_j - b_j \cos\alpha_j)Q_j + h_j \ , \qquad (4.13a)$$

and thus

$$p_j = c'(a_j \cos\alpha_j + b_j \sin\alpha_j) \ , \qquad (4.13b)$$

$$q_j = (c'/\rho)(a_j \sin\alpha_j - b_j \cos\alpha_j) \ , \qquad (4.13c)$$

$$h_j = p_j + h * (\text{equilibrium tide})_j \ . \qquad (4.13d)$$

We may insert a remark that if needed, a simple iterative process in A_j and ψ_j could be conceived along the following lines. From the initial values a_j and b_j, implied by the spherical-harmonic tidal coefficients, one computes

$$A_j = (a_j^2 + b_j^2)^{\frac{1}{2}}$$

and

$$\psi_j = \text{arc tg}(b_j/a_j) \ .$$

The adjusted values can then serve as new starting values denoted by primes:

$$A'_j = A_j(1+P_j) , \tag{4.14a}$$

$$\psi'_j = \psi_j + Q_j/\rho , \tag{4.14b}$$

from which the new starting values a'_j and b'_j are computed as

$$a'_j = A'_j \cos\psi'_j , \tag{4.15a}$$

$$b'_j = A'_j \sin\psi'_j . \tag{4.15b}$$

If the primes are omitted the adjustment process may start at this point, as if such initial values were directly available.

4.3 Solution

The observation equations consist of rows A_i, where i runs through all the observations, and of the corresponding constant terms L_i. The matrix of observation equations (A) consists of all the rows A_i and the vector of constant terms (L) consists of all the terms L_i. The altimeter observations are weighted independently; all of the A_i and L_i are normalized through the division by the appropriate sigma (square root of the variance). Thus the normal equations read

$$(A^T A)X + A^T L = 0 , \tag{4.16}$$

where X is the vector of all the parameters. The parameters involved in the short-arc adjustment of satellite altimetry are divided into three groups: 1) corrections to the spherical-harmonic potential coefficients, 2) tidal parameters and 3) corrections to the state vector parameters. The first group is envisioned as consisting of 225 parameters corresponding to a (14, 14) truncated spherical-harmonic model. The second group, absent in previous adjustments of satellite altimetry at AFGL, comprises the 18 parameters described earlier. The first two groups of 243 "terrestrial parameters" are assigned permanent storage in the computer core. The third group consists of six state vector parameters per (short) orbital arc; these parameters as well as the appropriate portions of normal equations are assigned reusable storage which is one of the main features of the short-arc algorithm described in detail in several AFGL reports. With the aid of this algorithm, the state vector parameters are eliminated from the normal equations and are solved for later (after the solution of the terrestrial parameters), arc by arc.

The row A_i in this study is composed of the "regular" entries associated with the original adjustment of satellite altimetry (without any tidal considerations), augmented by the 18 elements shown in (4.3). The value L_i is obtained by adding the value -N" for the observation point i to the "regular" value of the constant term, where

$$-N'' = -\sum_j h_j . \tag{4.17}$$

This can be expressed as

$$L_i = L_i' + L_i'' , \tag{4.18}$$

where L_i' represents the "regular" value of the constant term and L_i'' (computed through equation 4.17) is the contribution due to the considered tidal effects. The vector $A^T L$ in (4.16) is computed as

$$A^T L = \sum_i A_i^T L_i . \tag{4.19}$$

In the short-arc algorithn, this operation is performed for all the observations on one arc, after which the elimination of the pertinent six state vector parameters takes place. Subsequently, the same procedure takes place for the next arc, and so on.

At this juncture, a useful optional feature can be incorporated in the adjustment algorithm by simple means. One may ask, what would the vector in (4.19) become if all the L_i'' happened to be zero? Clearly, the answer is

$$A^T L' = \sum_i A_i^T L_i' . \tag{4.20}$$

One should note that no dimensions have changed, in other words, we have not reverted to the "regular" case without the tidal parameters. However, by virtue of the extra effort undertaken in (4.20) the computer program is able to solve an adjustment problem not only the "new" way with the tidal parameters, but also the "regular" way.

The key to this feature is the option to compute a "shadow" vector to $A_i^T L_i$. In particular, corresponding to each $A_i^T L_i$ there would be a "shadow" vector $A_i^T L_i'$ treated in exactly the same manner as the original vector, i.e., one would proceed to the summation (see 4.19 and 4.20), to the elimination of the state vector parameters, etc. The added tasks would in no way change the operations applied to the matrix of normal equations. The added computer burden in terms both of the run-time and storage requirements would thus be minimal. In order to obtain the solution corresponding to the absence of tidal parameters, the normal equations would be resolved using the final "shadow" vector in conjunction with very large weights (i.e., very small a priori sigmas) attributed to the tidal parameters. Since this "shadow" vector corresponds to the zero constant terms L_i'' and since the tidal parameters would be held essentially fixed at their initial values which are zero by construction, any tidal consideration would be virtually eliminated from the overall adjustment. It is to be noted that in a global adjustment of satellite altimetry, the formation of observation equations and the formation of normal equations together with the elimination of the state vector parameters require one or two orders of magnitude greater run-time than the actual solution of normal equations; once again, the computer burden needed in performing the added task would be minimal. The fruit of this undertaking consists in the possibility of comparing the results of a "new" adjustment (in which the tidal parameters are attributed realistic weights) with the results of a "regular" adjustment without tidal parameters. With the possible exception of the C_{20} coefficient which, if weakly weighted, could be influenced by the exclusion of the permanent tidal term, the adjusted spherical-harmonic potential coefficients should compare well between the two adjustments, especially if the altimeter data cover the oceans by a grid of passes distributed over a sufficiently long time interval. In the presence of tidal adjustment the r.m.s. residual is expected to be smaller -- by an amount which is of interest in itself -- than the corresponding r.m.s. residual in the absence of tidal adjustment. In both cases it is assumed that the (weighted) state vector parameters are uncontaminated by tidal effects; clearly, this assumption holds better with the tidal adjustment than without it.

In the closing paragraph of Section 4.2 the possibility of using an iterative scheme for tidal parameters was described. However, at the present such an approach is not envisioned for applications in real data reductions. First, in its original form the iterative process could be employed rigorously only if the tidal parameters were not weighted (or else changes would have to be introduced in the constant terms so that the weights would be associated with the initial values throughout the iterative process, not with the updated values). Second, even if a number of intermediate results were saved on tapes, the computer run-time would nevertheless increase significantly since in each iteration, all the steps associated with the tidal parameters and with the elimination of the state vector parameters (not to mention the solution of normal equations) would have to be repeated. Finally, since the tidal model is linear in tidal amplitudes and nonlinear in phase angles, an iterative process would be considered only if large corrections to the phase angles were suspected. However, there is little likelihood that this will occur on a global scale if reasonably good starting values of spherical-harmonic tidal coefficients are available; the likelihood would be greater for a regional adjustment. As a precautionary measure, one can ensure that these corrections are within acceptable limits by attributing relatively small sigmas (e.g., 5^0) to the Q_j parameters. On the other hand, corrections to the tidal amplitudes may be large. If desired, or if little *a priori* information exists, the P_j parameters can be weighted loosely without any ill effects, at least in theory. This property is especially useful with regard to the long-period constituents whose amplitude adjustment may shed light on the "empirical factor" e discussed earlier.

A few distinct situations associated with the weighting of tidal parameters can be discerned at once:

1) Large *a priori* sigmas (i.e., small weights) attributed to both kinds of parameters (P_j, Q_j). In this case, the amplitudes and the phase angles are essentially free to adjust. However, as cautioned above, the linearization of the tidal model could be compromised by large corrections to the phase angles. A narrower class of adjustment problems could be considered where the sigmas attributed to Q_j are no longer large.

2) Small *a priori* sigmas for P_j, Q_j. The amplitudes and the phase angles are essentially constant in the adjustment. This amounts to correcting the altimeter observations for the (presumably) known tidal effects and foregoing the tidal adjustment.

3) Small *a priori* sigmas for P_j, Q_j in conjunction with zero constituent heights. This corresponds to zero L_i'' values and to the "shadow" vector approach discussed previously, resulting in a "regular" adjustment with no tidal effects considered.

The above three groups are special. A usual tidal adjustment corresponds to "realistic" sigmas which could be envisioned as follows:

Long-period constituents ... $\sigma_{P_j} \simeq 1.$

This sigma is relatively large (a one-sigma correction would imply a magnitude of the amplitude itself), due to the uncertainty associated with the "empirical factor". Sigmas of 1.5 or 2 could also be adopted.

Dirunal and semidiurnal constituents ... $\sigma_{P_j} \simeq 0.5$, $\sigma_{Q_j} \simeq 10$.

The behavior of these constituents modeled through the spherical-harmonic tidal coefficients (as computed beforehand with the aid of Laplace Tidal Equations) is assumed to be reasonably well known. One could also consider $\sigma_{Q_j} \simeq 5$ (in ^{0}) which, in view of an earlier discussion, might offer some comfort with regard to the nonlinearity problem.

4.4 Practical Notes

Tidal spherical-harmonic expansion. As explained in the closing part of Section 3.2, a_j and b_j for the considered diurnal and semidiurnal constituents are obtained through a spherical-harmonic expansion with the tidal coefficients taken from [Estes, 1980]. These coefficients are complete through the degree and order (12,12). One thus has for the j-th constituent:

$$a_j = \sum_{n=0}^{12} \sum_{m=0}^{n} (a_{j_{nm}} \cos m\lambda + b_{j_{nm}} \sin m\lambda) P_{nm}(\sin\phi) , \tag{4.21a}$$

$$b_j = \sum_{n=0}^{12} \sum_{m=0}^{n} (c_{j_{nm}} \cos m\lambda + d_{j_{nm}} \sin m\lambda) P_{nm}(\sin\phi) ; \tag{4.21b}$$

the ocean tide for this constituent is then computed by (4.10'). In these equations, ϕ and λ are the known geocentric latitude and longitude, respectively, of the observation point associated with an event on a given satellite arc. When computing the "regular" geoid undulation in a global (spherical-harmonic) altimetric adjustment the following formula, adapted from equation (2.15) of [Blaha, 1979], is used:

$$\tilde{N} = r_0^* \sum_{n=2}^{14} (a/r')^n \sum_{m=0}^{n} (\Delta C_{nm} \cos m\lambda + \Delta S_{nm} \sin m\lambda) P_{nm}(\sin\phi) , \tag{4.22}$$

where ϕ and λ have the same interpretation as in (4.21 a,b) and where the other symbols, explained in Section 2.1 of the same reference, are of little consequence in this discussion. The similarity in form between (4.21 a,b) and (4.22) makes the computation of a_j and b_j a trivial task; the sines and cosines of the multiple angles $m\lambda$ as well as the associated Legendre functions $P_{nm}(\sin\phi)$ have all been computed (by recursive formulas) for the evaluation of (4.22). This makes the spherical-harmonic formulation of tidal effects particularly attractive in conjunction with a global adjustment of satellite altimetry.

Tidal arguments. The state vector parameters on each arc, stored on a magnetic tape, are preceded by the arc's ID. This ID gives the epoch time in terms of the day number (in 1978), hours and minutes (UT time). For all the individual events on the arc, the intervals $\Delta T = T_{\text{event}} - T_{\text{epoch}}$ are given in seconds and fractions thereof. The most economical approach to evaluating the needed Greenwich arguments is to evaluate them for T_{epoch} and to add a correction due to ΔT by multiplying this interval by the rate of change in α_j' (^{0}/second), gathered from Section 2.4. Thus we have

$$\alpha_{Mf}\,(\text{event}) = \alpha_{Mf}\,(\text{epoch}) + \Delta T * 0.00030501^0,$$

$$\alpha_{Mm}\,(\text{event}) = \alpha_{Mm}\,(\text{epoch}) + \Delta T * 0.00015122^0,$$

$$\alpha_{SSa}(\text{event}) = \alpha_{SSa}(\text{epoch}) + \Delta T * 0.00002282^0;$$

$$\alpha_{K_1}\,(\text{event}) = \alpha_{K_1}\,(\text{epoch}) + \Delta T * 0.00417807^0,$$

$$\alpha_{O_1}\,(\text{event}) = \alpha_{O_1}\,(\text{epoch}) + \Delta T * 0.00387307^0,$$

$$\alpha_{P_1}\,(\text{event}) = \alpha_{P_1}\,(\text{epoch}) + \Delta T * 0.00415526^0;$$

$$\alpha_{M_2}\text{ (event)} = \alpha_{M_2}\text{ (epoch)} + \Delta T * 0.00805114^{0} ,$$

$$\alpha_{S_2}\text{ (event)} = \alpha_{S_2}\text{ (epoch)} + \Delta T * 0.00833333^{0} ,$$

$$\alpha_{N_2}\text{ (event)} = \alpha_{N_2}\text{ (epoch)} + \Delta T * 0.00789992^{0} ,$$

$$\alpha_{K_2}\text{ (event)} = \alpha_{K_2}\text{ (epoch)} + \Delta T * 0.00835615^{0} .$$

The local arguments follow, as usual, upon adding λ to the Greenwich arguments for the diurnal constituents and 2λ for the semidiurnal constituents (nothing is added in case of the long-period constituents).

Regional tidal adjustment. Based on the residuals from a global adjustment of satellite altimetry, a regional adjustment in terms of point masses can be performed. This approach, described e.g. in the AFGL reports [Blaha, 1977, 1979, 1980], results in a more detailed local geoid. The point-mass magnitudes have been the only parameters present in the past "regular" approach to this second-phase adjustment where they have served in accommodating, in a least-squares sense, the first-phase altimeter residuals. The point-mass adjustment could be complemented by tidal parameters in a manner similar to that explained earlier. This possibility is now briefly outlined. Since the only role of the tidal parameters in such an adjustment is to "overcorrect" the values of P_j and Q_j from the first adjustment, the same coefficients in the observation equations can be used as those computed previously. And since these coefficients are stored on tape for each observation, augmenting the observation equations in the second adjustment is a trivial matter. These coefficients are applied with the positive sign because in the second adjustment they serve in accommodating the geoidal residuals (minus the altimeter residuals). The column-vector of parameters is then augmented by the 18 tidal parameters appearing in (4.4). The constant terms of observation equations, stored on tape, are simply minus the altimeter residuals from the first adjustment. The weights associated with the tidal parameters are suggested to be similar to those from the first adjustment, or perhaps greater since a general fit of the modeled sea surface to the observed sea surface has been accomplished and is not expected to change drastically in the second adjustment if the point-mass region is reasonably large. Due to the simple treatment of the tidal part in the point-mass adjustment, the run-time requirements are virtually unchanged from its "regular" counterpart.

5. CONCLUSION

In the recent past, SEASAT altimeter data together with spherical-harmonic potential coefficients (supplied by the GEM 10 model) and sets of state vector parameters (supplied by the NSWC precise ephemeris) have been adjusted at AFGL through the short-arc algorithm. In using several low degree and order truncations of the spherical-harmonic (S.H.) model, it has been observed that the empirical variance for geoid undulations is significantly lower than the theoretical variance. The details of this analysis can be found in Appendix 1, whose main findings are the following:

1) The SEASAT data, the ephemeris, the S.H. potential coefficients and the reference field parameters all appear to be of excellent quality;

2) The short-arc algorithm in conjunction with the seven-minute criterion described in previous reports does not introduce appreciable errors; and

3) Within the narrow band of frequencies considered (corresponding to the S.H. truncations between 8,8 and 16,16), the adjusted degree variances for geoid undulations reach only about 56% of their theoretical values.

During past efforts at AFGL, the altimeter residuals obtained from the global altimeter adjustment (with the S.H. potential coefficients and the state vector components as parameters) have served in regional modeling of short-wavelength geoidal features, as well as in studying geophysical phenomena such as ocean bottom topography. In this adjustment the geoid has been assumed to coincide with the ocean surface as sensored by the altimeter and the sea-surface effects have been ignored. However, due to increasing geophysical interest in a realistic representation of the open ocean tide, the latest development of the short-arc satellite altimetry model allows for the inclusion of the most important tidal constituents. In particular, an adjustment algorithm has been designed in which four long-period constituents, three diurnal constituents and four semidiurnal constituents may be subject to adjustment within the overall adjustment of SEASAT altimetric observations. Except for the long-period constituents where the phase angle is considered fixed, both tidal amplitude and phase angle are adjustable. This development is contained in the body of the present report.

One of the building blocks of the tidal adjustment just mentioned is the exploitation of the S.H. expansion of the constituent height. In particular, the S.H. tidal coefficients for the diurnal constituents K_1, O_1, P_1 and for the semidiurnal constituents M_2, S_2, N_2, K_2 can be adopted from [Estes, 1980], or from another reference on the subject, and can be treated with advantage by essentially the same algorithm which serves in the computation of geoid undulations from the S.H. potential coefficients. One has to keep in mind, however, that very serious errors would occur if the tidal coefficients were truncated (the above reference supplies a set of 12,12 coefficients for each of the seven constituents). This property stems from the method used in the determination of these coefficients, in which the land areas were ignored (the solution was not held to zero over land). Thus, as stated on page 80 of [Estes, 1980], "completely spurious values are obtained if the expansion is evaluated over land areas." This has been confirmed by evaluating a few constituent heights. For example, although for the middle of the U.S. none of the constituent heights exhibit an unreasonable magnitude, the M_2 tide for the middle of Asia (latitude 45°, longitude 90°) at a given time epoch (day 200 in 1978, UT = 2 hours) is computed as 7.1m.

In addition to exercising proper caution when dealing with the tidal coefficients, one should view the individual tidal constituents in relation to the specific orbital characteristics of the satellite used in data acquisition. In particular, the constituents which cannot be separated from others by the altimeter data should be adequately modeled and enforced a priori. If left to the adjustment, such constituents would result in aliasing tidal constituents of different frequencies. For example, if a satellite performing altimeter measurements were sun-synchronous (this is not the case with SEASAT), the constituents S_1 and S_2 would be aliased into the zero frequency, i.e., this particular satellite, sampling theoretically only the high solar tide, would sense them as a constant constituent. In general, then, one is faced with the necessity of attributing a higher weight to the parameters (here the tidal amplitude and the phase angle) of each tidal constituent which cannot be satisfactorily resolved from the altimetry.

In the present adjustment model of SEASAT altimetry and the selected tidal effects, the constituents to be treated with care are K_1, P_1, S_2 and K_2. As stated in [TOPEX, 1981], the diurnal constituents K_1 and P_1 are aliased to a six-month and a constant constituent, respectively; the semidiurnal constituents S_2 and K_2 are similarly aliased to a six-month and a three-month constituent. Since the useful life span of SEASAT amounted to only about three months, most of these constituents cannot be properly resolved, as is already the case with SSa (the semiannual constituent). To further complicate the matters, K_1 and S_2, if not partially or totally constrained by appropriate weighting, would be aliased into this SSa. Accordingly, it may be necessary to lower the *a priori* sigmas associated with the eight parameters corresponding to the above four constituents (K_1, P_1, S_2 and K_2, two parameters per constituent), perhaps to ¼ or less of the sigmas associated with the other diurnal and semidiurnal constituents.

APPENDIX 1

ANALYSIS OF LOW DEGREE VARIANCES FOR GEOID UNDULATIONS USING SEASAT ALTIMETER DATA

The global adjustment of SEASAT altimeter data yields, at an initial stage, the misclosures in the altimeter observation equations formulated in terms of spherical-harmonic (S.H.) potential coefficients and state vector parameters (six per short orbital arc). These misclosures are also called the constant terms of observation equations. They reflect on the errors in the altimeter measurements (the noise of the system), in the a priori state vector parameters, in the short-arc algorithm (a modeling error), and in the input S.H. potential coefficients and reference field parameters. But, most of all, they reflect on the geoidal detail ignored by the adjustment. Such detail depends on the truncation of the S.H. model and can be represented by a variance, called here the "theoretical variance", obtained as a sum of "theoretical degree variances" for geoid undulations, where the summation extends over all the neglected degrees. Thus, in a model truncated at degree and order (14,14) the summation extends over the degrees 15 through perhaps 1,000 (beyond this degree the contribution is negligible for all practical purposes). The theoretical degree variances are in turn computed from the covariance function.

When an actual adjustment with SEASAT data was carried out, it was observed that the root mean square (rms) of the altimetric misclosures in the (14,14) truncated S.H. model was significantly lower than the "theoretical sigma" (the square root of the theoretical variance). Yet, this rms contains error contributions from the other sources of information mentioned above, i.e., from the ephemeris, the S.H. potential coefficients together with the reference field parameters, and the SEASAT altimetry. The low rms value serves as an indication of excellent quality of these sources and, in addition, as an indication of sufficient accuracy in the short-arc algorithm applied in conjunction with the seven-minute arc criterion discussed in previous reports. This is supported by the fact that the average misclosure is nearly zero (-0.1m). The rms value suggests that the theoretical formula for the covariance function may be too conservative, at least insofar as the geoid undulations for relatively low degree and order truncations are concerned.

In analyzing this phenomenon the variance due to the ephemeris, in particular, due to the radial component of the state vector parameters, has been subtracted from the mean square of the misclosures yielding the "empirical variance". This implies that the other variances affecting the rms (the variances due to the input S.H. coefficients and the reference field parameters, to the altimeter noise, and to the short-arc algorithm) have been assumed to be negligible; otherwise the already low empirical variance would have been even lower. The adopted procedure for computing this variance is symbolically expressed as

$$\hat{\sigma}^2 \simeq \mathrm{rms}^2(\mathrm{misclosures}) - \sigma^2(\mathrm{ephemeris}) \ , \qquad \text{(A1.1)}$$

where "^" represents "empirical". An initial step in the misclosure analysis was undertaken in Appendix 1 of [Blaha, 1981] where the sources of information beyond the SEASAT altimeter were identified (GEM 10 for the S.H. potential coefficients, NSWC precise ephemeris for the state vector parameters, and

IUGG/IAG, 1975, for the reference gravity field parameters). The S.H. model considered therein was limited to the (14,14) subset. However, the global rms misclosures have been since computed also in conjunction with the (8,8), (10,10), (12,12), and (16,16) truncated models. In order to assess the usefulness of these results for further analysis, the topic of the theoretical variance will be addressed in more detail.

If an adjustment process is carried out in terms of an (n,n) S.H. model, the theoretical variance for geoid undulations characterizing the higher frequency content ignored by the adjustment is obtained as a summation of theoretical degree variances from degree n+1 onward. This variance is expressed as

$$\sigma^2_{n+1,\infty} = \sum_{k=n+1}^{\infty} \sigma_k^2 ,$$

where σ_k^2 is the k-th degree theoretical variance for geoid undulations. The symbol ∞ is replaced in practice by a suitable large number such as 1,000. According to the formulas in [Tscherning and Rapp, 1974] the theoretical k-th degree variance for geoid undulations (σ_k^2) is computed from the theoretical k-th degree variance for gravity anomalies ($\tilde{\sigma}_k^2$) which is in turn obtained from the covariance function, as follows:

$$\sigma_k^2 = R^2\, \tilde{\sigma}_k^2/[G^2(k-1)^2] ,$$

$$\tilde{\sigma}_k^2 = s^{k+2} A(k-1)/[(k-2)(k+B)] ,$$

where G is the average value of gravity (979.8 gal), R is the earth's mean radius (6,371 km), and

$$s = 0.999617 ,$$

$$A = 425.28 \text{ mgal}^2 ,$$

$$B = 24, \text{ an integer} .$$

These formulas have been adapted from equations (10), (25A) and from Table 7 of the same reference.

Upon carrying out the operations indicated above, the theoretical k-th degree variance for geoid undulations reads

$$\sigma_k^2 = 0.999617^{k+2} \; 17{,}981 \text{m}^2/[(k-1)(k-2)(k+24)] . \qquad \text{(A1.2)}$$

Upon summing up these variances from the chosen degree on, one obtains the results presented in the second column of Table 1. The "n+1" in the first column implies the (n,n) truncated S.H. model and indicates that the geoidal variances (both theoretical and empirical) are due to the neglected degrees, from n+1 onward. The empirical variances are featured in the third column. They are computed according to (A1.1), where the rms misclosures are obtained from the altimeter observation equations using the (n,n) truncated model, and where

$$\sigma(\text{ephemeris}) = 1.6 \text{ m} ,$$

which is the *a priori* standard error in the radial component of the state vector parameters. It is apparent from Table 1 that the empirical variance for the low degree truncations considered is significantly lower than its theoretical counterpart. For the truncation (8,8) the ratio of these two variances is 0.4884, for the truncation (10,10) it recedes to 0.4004 and, eventually, for the truncation (16,16) it rises to 0.4917. Since a similar pattern can be assumed also for the intermediate truncations, one can conclude that for any truncation between (8,8) and (16,16) this ratio maintains itself under 0.5. However, it would most likely climb above 0.5 already for the truncation (17,17) due to the rising uncertainty in the S.H. potential coefficients with the increasing degrees. In accordance with the outlined procedure, this effect should ideally be eliminated from the empirical variance (the variance due to the input S.H. coefficients was previously assumed to be negligible).

n+1	$\sigma^2_{n+1,\infty}$	$\hat{\sigma}^2_{n+1,\infty}$
8+1	57.758	28.21
10+1	40.758	16.32
12+1	30.562	12.88
14+1	23.886	10.13
16+1	19.243	9.46

Table 1

Theoretical variances (in m^2) and empirical variances (in m^2) for geoid undulations corresponding to selected degree and order truncations (n,n)

The marginal usefulness of the S.H. model beyond (14,14) for the study of the empirical variances is corroborated by the statement on page 19 of [Khan, 1981]: "At this stage it is generally believed that the long wavelength components of the gravity field are well-determined to n=14". Figure 2 of this reference, showing the approximate level of accuracy as a function of frequency, quantifies this statement for the GEM 10B coefficients through the degree n=22 by showing that for n=14 the "accuracy determination" is 54% while for n=16 it is only about 33% and it continues to diminish as n increases. When faced with these facts one could choose to disregard the empirical variance for n=16 and perhaps even for n=14, and use only the first three empirical variances in Table 1. However, in considering that only the differences between the empirical variances will eventually be used as observables in evaluating the formula (A1.2), it becomes clear that such an approach would drastically reduce the already small number of available observations.

Another avenue one could take in analyzing (A1.2) would be to use all of the empirical variances listed in Table 1 and to correct them for the effect of uncertainty in the input S.H. coefficients. However, several difficulties would complicate this approach:

a) The accuracy evaluation in Figure 2 of [Khan, 1981] concerns the GEM 10B coefficients and not the GEM 10 coefficients which have served in the computation of empirical variances from SEASAT altimetry;

b) For a given degree k, the "accuracy determination", y_k%, is not specific enough to be unequivocally applicable to the problem at hand. In particular, it is not clear whether the effect of the uncertainty in the k-th degree S.H. coefficients is exactly $(1-y_k)$% of the theoretical degree variance σ_k^2, or some other percentage; in other words, it is not clear what probability statement could be associated with the uncertainty $(1-y_k)\sigma_k^2$;

c) Even if $(1-y_k)\sigma_k^2$ were known to be the theoretical effect of this uncertainty one would suspect that $(1-y_k)t\ \sigma_k^2$, rather than $(1-y_k)\sigma_k^2$, should be subtracted from each uncorrected empirical degree variance; here t is some reducing factor due to the smaller magnitudes of the empirical variances as compared to their theoretical counterparts; and

d) If an adjustment model for degree variances should be represented by the formula (A1.2) or its equivalent, and if the observables should be represented by the corrected empirical degree variances or their combinations, one would still be faced with the task of attributing proper weights to the observations in order to carry out a reasonably rigorous least-squares adjustment process; for the sake of simplicity (but without any further justification), one might wish to adopt a unit weight matrix for this purpose, but the variance-covariance matrices after adjustment would be difficult to interpret in a meaningful manner.

In the approach adopted herein the empirical degree variances are not corrected in any way, but the uncertainty in the S.H. coefficients serves in the establishment of a weighting scheme in the least-squares adjustment with (A1.2) as the basic model. The latter is rewritten as

$$\sigma_k^2 = s^{k+2} C/[(k-1)(k-2)(k+B)] , \tag{A1.3}$$

where the parameters are s, C and B. The initial values of these parameters were presented earlier as

$$s = 0.999617 , \tag{A1.4a}$$

$$C = 17{,}981 \text{ m}^2 , \tag{A1.4b}$$

$$B = 24. \tag{A1.4c}$$

Since various approximations applied to a weighting scheme are much less dangerous than the same kind of approximations carried into the actual observations, the difficulties encountered in the preceding paragraph are now unimportant. By the same token, the association of the measure of uncertainty, $(1-y_k)\sigma_k^2$, with the standard error (sigma) of the quantity σ_k^2 would be acceptable if this standard error were indeed necessary. However, if the actual observations are the empirical degree variances ($\hat{\sigma}_k^2$) which, for the degrees involved, are systematically lower than the theoretical degree variances, one can take as the a priori standard error:

$$\text{sigma}\ (\hat{\sigma}_k^2) = (1-y_k)t\ \sigma_k^2\ , \tag{A1.5}$$

where the reducing factor t is computed as the ratio between the sum of the pertinent empirical degree variances (here between the degrees 9 and 16) and the sum of the corresponding theoretical degree variances,

$$t = \hat{\sigma}^2_{9,16} \,/\, \sigma^2_{9,16}\ .$$

The degree variances have been limited to n=9 through n=16 for practical reasons. As stated in [Blaha, 1981], the errors introduced in the short-arc algorithm due to the degree and order truncation below (8,8) cannot be tolerated in SEASAT altimetry. In fact, even the truncations (8,8) and (9,9) introduce detectable (but tolerable) errors in the satellite positions, but the set (10,10) is already completely satisfactory. It should be noted that these considerations are valid only if the arc's duration does not surpass 7 minutes and if the state vector parameters are given at, or very nearly at, the mid-arc. For these reasons the empirical variances in

Table 1 are listed starting with n+1 = 8+1. On the other hand, the last value listed corresponds to n+1 = 16+1, due to the large uncertainty in the S.H. coefficients discussed earlier. Clearly, with the observations available only for the degrees indicated, the results of the present analysis are applicable only to the low-frequency range of the S.H. model, between n=9 and n=16. Since the desired combinations of degree variances are

$$\sigma^2_{9,16} \equiv \sum_{k=9}^{16} \sigma^2_k = \sigma^2_{9,\infty} - \sigma^2_{17,\infty} ,$$

$$\hat{\sigma}^2_{9,16} \equiv \sum_{k=9}^{16} \hat{\sigma}^2_k = \hat{\sigma}^2_{9,\infty} - \hat{\sigma}^2_{17,\infty} ,$$

Table 1 yields

$$t = (28.21\,m^2 - 9.46\,m^2)/(57.758\,m^2 - 19.243\,m^2) = 0.4868 . \qquad \text{(A1.6)}$$

The variance of the quantity $\hat{\sigma}^2_k$ is computed from (A1.5) as

$$\text{var}(\hat{\sigma}^2_k) = [(1-y_k)t\ \sigma^2_k]^2 . \qquad \text{(A1.7)}$$

For the weighting purposes, the empirical variances for any degrees are considered mutually independent. Since the empirical variances will be considered in combinations (in fact, they will be added pair-wise, each pair forming one observation), one has

$$\text{var}(\hat{\sigma}^2_{n_1,n_2}) = \sum_{k=n_1}^{n_2} \text{var}(\hat{\sigma}^2_k) . \qquad \text{(A1.8)}$$

The diagonal weight matrix will thus be composed of the reciprocal values in (A1.8). Although these variances are only very approximate -- and will be further simplified below -- their magnitudes are realistic and allow meaningful estimates of the variance-covariance matrices after adjustment.

Based on the foregoing discussion, one can compute the desired weights upon adopting the accuracy determination factor (denoted here as y_k%) from Figure 2 of [Khan, 1981]. This factor is listed, in the form $(1-y_k)$%, in the third column of Table 2 for k between 9 and 16. In the cases k=9 and k=10 the values $(1-y_k)$% are augmented by 7% and 3%, respectively, due to the errors introduced in the short-arc algorithm via the truncation (8,8). Although these percentages would be far too high for the present configuration of SEASAT short arcs, a conservative approach is warranted because the pertinent rms misclosures were computed before the completion of the new short-arc preprocessor which ensures that the epoch is almost exactly at mid-arc. The fourth column of Table 2 gives the variances of $\hat{\sigma}_k^2$ according to (A1.7), with t taken from (A1.6). Since each of the observations will consist of two empirical variances added together, the observational variances are formed according to (A1.8), where $n_2=n_1+1$, and are listed, for the degrees 9 and 10, 11 and 12, 13 and 14, 15 and 16, in the fifth column of Table 2. It is apparent that these variances (in m^4) are fairly close to unity. Since only approximate weights are needed and since several approximations in their formation have already been introduced, little damage will be done if the weight matrix is adopted as the unit matrix.

k	σ_k^2	$(1-y_k)\%$	$\mathrm{var}(\hat{\sigma}_k^2)$	$\mathrm{var}(\hat{\sigma}_{n1,n2}^2)$
9	9.689	5%+7%=12%	0.3203	0.61
10	7.311	12%+3%=15%	0.2850	
11	5.680	19%	0.2760	0.63
12	4.516	27%	0.3523	
13	3.661	36%	0.4116	0.87
14	3.015	46%	0.4558	
15	2.517	57%	0.4878	0.97
16	2.126	67%	0.4808	

Table 2

Variances (in m^4) of the empirical degree variances and of their pair-wise combinations

As has already been indicated, each observation equation corresponds to two degree variances. The constant terms in these equations are formed as the sum of two theoretical degree variances minus the sum of two corresponding empirical degree variances. The pair-wise sums of the theoretical degree variances are apparent from the second column of Table 2. They can also be obtained from the second column of Table 1 as the differences between the appropriate theoretical variances; for example, $\sigma_{9,10}^2$ is obtained (in m^2) from Table 2 as 9.689 + 7.311 = 17.000 or from Table 1 as 57.758 - 40.758 = 17.000. The corresponding combinations of the empirical degree variances are apparent from the third column of Table 1, for example, $\hat{\sigma}_{9,10}^2$ = 28.21 - 16.32 = 11.89 (in m^2). Table 3 lists the pertinent sums of theoretical and empirical degree variances (columns two and three, respectively), and of their differences under the heading L_{n_1,n_2}. The totals for columns two and three correspond to the differences "top-bottom" in the same columns of Table 1.

n_1, n_2	$\sigma^2_{n_1,n_2}$	$\hat{\sigma}^2_{n_1,n_2}$	L_{n_1,n_2}
9,10	17.000	11.89	5.110
11,12	10.196	3.44	6.756
13,14	6.676	2.75	3.926
15,16	4.643	0.67	3.973
Σ	38.515	18.75	19.765

Table 3

Constant terms L_{n_1,n_2} (in m^2) in the observation equations

The matrix of observation equations in the case of individual degree variances (see A1.3) would contain the partial derivatives of σ_k^2 with respect to s, C and B:

$$\partial\sigma_k^2/\partial s = \sigma_k^2(k+2)/s \, ,$$

$$\partial\sigma_k^2/\partial C = \sigma_k^2/C \, ,$$

$$\partial\sigma_k^2/\partial B = -\sigma_k^2/(k+B) \, .$$

The observation equations for the sums of degree variances read

$$V_{n_1,n_2} = [\sum_{k=n_1}^{n_2} \sigma_k^2(k+2)/s]\,ds + [\sum_{k=n_1}^{n_2} \sigma_k^2/C]\,dC - [\sum_{k=n_1}^{n_2} \sigma_k^2/(k+B)]\,dB + L_{n_1,n_2} .$$

In order to make the numbers in the matrix of observation equation (A) comparable, one may replace ds, dC and dB by ds', dC' and dB' in such a way that

$$ds = (1/100)\, ds' , \tag{A1.9a}$$

$$dC = (C/10)\, dC' , \tag{A1.9b}$$

$$dB = -4dB' ; \tag{A1.9c}$$

the observation equations then become

$$\begin{aligned} V_{n_1,n_2} = {} & [\sum_{k=n_1}^{n_2} 0.01\, \sigma_k^2(k+2)/s]\,ds' + [\sum_{k=n_1}^{n_2} 0.1\, \sigma_k^2]\,dC' \\ & + [\sum_{k=n_1}^{n_2} 4\sigma_k^2/(k+B)]\,dB' + L_{n_1,n_2} . \end{aligned} \tag{A1.10}$$

The elements of the matrix A formed from (A1.10) are computed in a straightforward fashion using the values in (A1.4 a-c) and σ_k^2 from the second column of Table 2 (the additions in A1.10 are pair-wise as before):

$$A = \begin{bmatrix} 1.9438 & 1.7000 & 2.0345 \\ 1.3712 & 1.0196 & 1.1509 \\ 1.0320 & 0.6676 & 0.7132 \\ 0.8109 & 0.4643 & 0.4708 \end{bmatrix} . \tag{A1.11}$$

The observation equations are written in the matrix form as

$$V = AX + L ,$$

where V is the vector of (four) residuals, X is the vector of (three) parameters, ds', dC' and dB', and L is the vector of (four) constant terms as presented in Table 3.

Since the weight matrix has been stipulated to be

$$P = I ,$$

where the units are m^{-4}, the least-squares solution is given by

$$X = -N^{-1} A^T L ,$$

$$N = A^T A ,$$

and the variance-covariance matrix for the parameters X is

$$C_X = N^{-1} .$$

The original parameters (denoted by Y) are computed as in (A1.9 a-c). From the general relation

$$Y = GX ,$$

$$C_Y = G C_X G^T ,$$

it follows for the specific linear functions, such as

$$y_1 = c_1 x_1 ,$$

$$y_2 = c_2 x_2 , \dots$$

that

$$\sigma_{y_1} = |c_1|\, \sigma_{x_1} ,$$

$$\sigma_{y_2} \quad |c_2|\, \sigma_{x_2} , \ldots$$

and

$$\rho_{y_1,y_2} = \text{sign}\ (c_1 c_2)\ \rho_{x_1,x_2} , \ldots$$

yielding the sigmas and the correlation coefficients (ρ) for the original parameters ds, dC and dB.

As may have been anticipated already from (A1.11) showing that the values in all three columns of A behave in a quite similar manner, the matrix of normal equations (N) is severely ill-conditioned. In fact, both X and C_X contain exceedingly large and unrealistic values. The matrix of correlation coefficients, formed from N^{-1} and modified so that the signs correspond to the original parameters, reads

$$\rho = \begin{bmatrix} 1. & -0.999079 & -0.998350 \\ \text{symm.} & 1. & +0.999892 \\ & & 1. \end{bmatrix} . \qquad \text{(A1.12)}$$

Although the results of this adjustment are meaningless, the rms residual has been computed for comparison purposes as

$$rms_V = 0.668\, m^2 .$$

This value would be larger if dB were rounded to the nearest integer.

Clearly, all three parameters cannot be determined in any meaningful way from the observations that encompass only the degrees $n=9$ to $n=16$. One could attempt to adjust only two parameters while holding the third fixed (the corresponding column in A would be disregarded). However, since any two of the three columns in (A1.11) are quite comparable, ill-conditioning is again a problem. For example, in terms of the original parameters s, C one obtains

$$ds = -0.09618 \,, \qquad \sigma_s = 0.02730 \,,$$
$$dC = +13.880\ m^2 \,, \qquad \sigma_C = 6{,}225\ m^2 \,.$$

Both the corrections and the sigmas are quite large. The post-adjustment correlation coefficient, although lower than any one in (A1.12), is still very large, namely

$$\rho_{sC} = -0.990871 \,.$$

the residual fit is necessarily worse than previously (with three parameters), represented by

$$rms_V = 0.874\ m^2 \,.$$

The adjustments of the other two pairs of parameters is equally unimpressive; in fact, the correlation coefficient for dC and dB reaches +0.998928.

Another possibility of adjusting two parameters is to restrict the model to only one adjustable parameter (s, C or B) and to add a linear parameter, D. The following results are obtained for C and D:

$$dC = -2{,}058 \text{ m}^2 , \qquad \sigma_C = 1{,}914 \text{ m}^2 ,$$

$$D = -1.920 \text{ m}^2 , \qquad \sigma_D = 0.570 \text{ m}^2 ;$$

$$\rho_{CD} = -0.899 ;$$

$$rms_V = 1.017 \text{ m}^2 .$$

Comparable results would be obtained also if dC were replaced by ds or dB (if the latter were rounded to the nearest integer the rms residual would necessarily increase). However, it is not the purpose of the present analysis to change the structure of the degree variance model. Accordingly, this example has served merely as an illustration of a less ill-conditioned system (on the other hand, the rms residual has increased somewhat).

It has been already noticed that due to the increased uncertainty in the S.H. potential coefficients, the truncation (16,16) may be of only marginal value. For this reason another series of adjustments has been carried out, but without the last observation equation. All the results, including the goodness of fit represented here by the rms residual, have suffered when compared to their counterparts with four (rather than three) observation equations. The worsening is most noticeable in the correlation coefficients. A certain amount of similarity has been observed only when adjusting one parameter alone. The results of the one-parameter adjustment will now be presented for the original four observation equations. With s as the parameter, one obtains

$$ds = -0.035862 , \qquad \sigma_s = 0.003681 ;$$

$$rms_V = 1.417 \text{ m}^2 .$$

By comparison, the adjustment of B yields

$$dB = +14.751 , \qquad \sigma_B = 1.607;$$

$$rms_V = 2.163 \text{ m}^2 .$$

This residual fit would be even worse if dB were rounded to 15. One notices that the adjusted parameter $B^a = B + dB$ would be 39 which is greatly, perhaps unrealistically, removed from the theoretical value $B = 24$.

The most useful in the series of adjustments carried out in the course of this analysis is probably the adjustment of the single parameter C. Upon obtaining the adjusted parameter $C^a = C + dC$, one can construct the ratio C^a/C as a simple measure of judging the decrease in degree variances, theoretical versus empirical, within the range n=9 and n=16. The results obtained with this model are

$$dC = -7{,}849 \text{ m}^2 , \qquad \sigma_C = 839 \text{ m}^2 ;$$

$$C^a = 10{,}132 \text{ m}^2 , \tag{A1.13a}$$

$$C^a/C = 0.5635 . \tag{A1.13b}$$

The result (A1.13b) indicates that within the narrow band of the S.H. frequencies treated herein the adjusted degree variances would reach only about 56% of their theoretical value (the adjusted sigmas would accordingly reach about 75% of their theoretical value). The rms residual is quite high in this case, namely

$$rms_V = 1.966 \text{ m}^2 .$$

This suggests that due to the unaccounted for errors in the S.H. potential coefficients and perhaps to other causes, the empirical degree variances do not behave exactly according to the model, the correcting factor C^a/C notwithstanding. But an important conclusion that can be reached is that in spite of these errors the empirical variances for the considered (low)

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frequencies are substantially smaller than their theoretical counterparts, and that the ratio of these two kinds of variances is approximately 0.56.

APPENDIX 2

EXPANSIONS IN SURFACE SPHERICAL-HARMONICS

Most of the formulas used in this appendix can be found, in one form or another, in the standard literature and will thus be stated without proof or reference. In the derivations, use will be made of the spherical coordinates θ (colatitude) and λ (longitude).

A function on the sphere, $f(\theta,\lambda)$, can be expanded in surface spherical-harmonics:

$$f(\theta,\lambda) = \sum_{n=0}^{\infty} f_n(\theta,\lambda) , \tag{A2.1}$$

where the surface spherical-harmonic of n-th degree is

$$f_n(\theta,\lambda) = \sum_{m=0}^{n} [a_{nm} R_{nm}(\theta,\lambda) + b_{nm} S_{nm}(\theta,\lambda)] , \tag{A2.2}$$

a's and b's being the coefficients, and R's and S's being defined as

$$R_{no}(\theta,\lambda) = P_{no}(\cos\theta) \equiv P_n(\cos\theta) , \qquad S_{no}(\theta,\lambda) = 0 ,$$

$$R_{nm}(\theta,\lambda) = P_{nm}(\cos\theta) \cos m\lambda , \qquad m > 0 ,$$

$$S_{nm}(\theta,\lambda) = P_{nm}(\cos\theta) \sin m\lambda , \qquad m > 0 ,$$

where $P_{nm}(\cos\theta)$ are the associated Legendre functions in the argument $\cos\theta$. Equation (A2.1) can thus be rewritten in a familiar form as

$$\begin{aligned} f(\theta,\lambda) &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} [a_{nm} R_{nm}(\theta,\lambda) + b_{nm} S_{nm}(\theta,\lambda)] \\ &\equiv \sum_{n=0}^{\infty} \sum_{m=0}^{n} (a_{nm} \cos m\lambda + b_{nm} \sin m\lambda) P_{nm}(\cos\theta) . \end{aligned} \tag{A2.3}$$

If the spherical angle ψ separates two points with the spherical coordinates (θ,λ) and $(\bar{\theta},\bar{\lambda})$, the following "addition formula" applies:

$$P_n(\cos\psi) = P_n(\cos\theta)\, P_n(\cos\bar{\theta}) + 2 \sum_{m=1}^{n} [(n-m)!/(n+m)!]\,[R_{nm}(\theta,\lambda)\, R_{nm}(\bar{\theta},\bar{\lambda}) + S_{nm}(\theta,\lambda)\, S_{nm}(\bar{\theta},\bar{\lambda})] \quad , \tag{A2.4}$$

where

$$\cos\psi = \cos\theta \cos\bar{\theta} + \sin\theta \sin\bar{\theta} \cos(\bar{\lambda} - \lambda) \ .$$

Next, the orthogonality relations are presented:

$$\iint_\Omega [R_{no}(\theta,\lambda)]^2 d\Omega = 4\pi/(2n+1) \ , \tag{A2.5a}$$

$$\iint_\Omega [R_{nm}(\theta,\lambda)]^2 d\Omega = \iint_\Omega [S_{nm}(\theta,\lambda)]^2 d\Omega = [2\pi/(2n+1)]\,[(n+m)!/(n-m)!] \ , \quad m > 0 \ , \tag{A2.5b}$$

where $d\Omega$ is the solid angle element and Ω represents the surface of a unit sphere. It is important to note that the integrals of all the other possible products not covered by (A2.5 a,b) vanish; one type of such products is $R_{nm}(\theta,\lambda)\ R_{k\ell}(\theta,\lambda)$, $k \neq n$ and/or $\ell \neq m$, where the R's could be replaced by the S's, and another type is $R_{nm}(\theta,\lambda)\ S_{k\ell}(\theta,\lambda)$ for any n, m, k, ℓ.

The relation (A2.3) leads to the statement

$$I(\theta,\lambda) \equiv \iint_\Omega f(\bar{\theta},\bar{\lambda})\, P_n(\cos\psi)\, d\bar{\Omega} = \iint_\Omega \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} [a_{k\ell}\, R_{k\ell}(\bar{\theta},\bar{\lambda}) + b_{k\ell}\, S_{k\ell}(\bar{\theta},\bar{\lambda})] \cdot P_n(\cos\psi)\, d\bar{\Omega} \ ,$$

associated, as indicated, with the (fixed) point (θ,λ). Upon expanding $P_n(\cos\psi)$ according to (A2.4) and utilizing the orthogonality relations, one obtains

$$I(\theta,\lambda) = R_{no}(\theta,\lambda)\ a_{no}[4\pi/(2n+1)] + 2 \sum_{m=1}^{n} [(n-m)!/(n+m)!]$$

$$* \{R_{nm}(\theta,\lambda)\ a_{nm}[2\pi/(2n+1)][(n+m)!/(n-m)!] + S_{nm}(\theta,\lambda)\ b_{nm}$$
$$* [2\pi/(2n+1)][(n+m)!/(n-m)!]\} .$$

This last result is written at once as

$$I(\theta,\lambda) = [4\pi/(2n+1)] \sum_{m=0}^{n} [a_{nm}\ R_{nm}(\theta,\lambda) + b_{nm}\ S_{nm}(\theta,\lambda)] ,$$

which, with the aid of (A2.2), yields the desired relation:

$$\iint_{\Omega} f(\bar{\theta},\bar{\lambda})\ P_n(\cos\psi)\ d\bar{\Omega} = [4\pi/(2n+1)] f_n(\theta,\lambda) . \qquad \text{(A2.6)}$$

REFERENCES

Blaha, G., The Combination of Gravity and Satellite Altimetry Data for Determining the Geoid Surface. AFCRL Report No. 75-0347, Air Force Cambridge Research Laboratories, Hanscom AFB, Bedford, Massachusetts, 1975.

Blaha, G., Refinements in the Combined Adjustment of Satellite Altimetry and Gravity Anomaly Data. AFGL Technical Report No. 77-0169, Air Force Geophysics Laboratory, Hanscom AFB, Massachusetts, 1977.

Blaha, G., "Refinement of the Short Arc Satellite Altimetry Adjustment Model". Paper published in Bulletin Géodésique, Vol. 51, No. 1, Bureau Central de l'Association Internationale de Géodésie, Paris, France, 1977'.

Blaha, G., Improved Determinations of the Earth's Gravity Field. AFGL Technical Report No. 79-0058, Air Force Geophysics Laboratory, Hanscom AFB, Massachusetts, 1979.

Blaha, G., Extended Applicability of the Spherical-Harmonic and Point-Mass Modeling of the Gravity Field. AFGL Technical Report No. 80-0180, Air Force Geophysics Laboratory, Hanscom AFB, Massachusetts, 1980.

Blaha, G., SEASAT Altimetry Adjustment Model Including Tidal and Other Sea Surface Effects. AFGL Technical Report No. 81-0152, Air Force Geophysics Laboratory, Hanscom AFB, Massachusetts, 1981.

Estes, R.H., A Computer Software System for the Generation of Global Ocean Tides Including Self-Gravitation and Crustal Loading Effects. GSFC Report X-920-77-82, Goddard Space Flight Center, Greenbelt, Maryland, 1977.

Estes, R.H., "A Simulation of Global Ocean Tide Recovery Using Altimeter Data with Systematic Orbit Error". Paper published in Marine Geodesy, Volume 3, Nos. 1-4, Crane Russak, New York, 1980.

Godin, G., The Analysis of Tides. University of Toronto Press, Toronto, Ontario, Canada, 1972.

Khan, M.A., Satellite Contributions to Geophysical Exploration at Sea. Hawaii Institute of Geophysics contribution No. 1178, University of Hawaii, Honolulu, 1981.

Lisitzin, E., Sea-Level Changes. American Elsevier Publishing Co., Inc., New York, 1974.

Parke, M.E., and M.C. Hendershott, "M2, S2, K1 Models of the Global Ocean Tide on an Elastic Earth". Paper published in Marine Geodesy, Volume 3, Nos. 1-4, Crane Russak, New York, 1980.

Schwiderski, E.W., "Ocean Tides, Part I: Global Ocean Tidal Equations". Paper published in Marine Geodesy, Volume 3, Nos. 1-4, Crane Russak, New York, 1980.

TOPEX Science Working Group, Satellite Altimetric Measurements of the Ocean. NASA, Jet Propulsion Laboratory of the California Institute of Technology, Pasadena, March 1, 1981.

Tscherning, C.C. and R.H. Rapp, Closed Covariance Expressions for Gravity Anomalies, Geoid Undulations, and Deflections of the Vertical Implied by Anomaly Degree Variance Models. Report No. 208, Department of Geodetic Science, The Ohio State University, Columbus, 1974.

USCGS, Manual of Harmonic Analysis and Prediction of Tides. Special Publication No. 98, U.S. Department of Commerce, U.S. Coast and Geodetic Survey (USCGS), Reprinted with corrections, 1958.

Vaníček, P., Tidal Corrections to Geodetic Quantities. NOAA Technical Report NOS 83 NGS 14, U.S. Department of Commerce, NOAA, NOS, Rockville, Maryland, 1980.